Quantum Field Theory (Quantum Electrodynamics)

Problem Set 7 2 & 4 December 2024

1. Quantization of the Dirac Field

The mode expansion of the Dirac field can be written in terms of ladder operators as

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \sum_i \left(u_i(\vec{p}) a_i(\vec{p}) e^{-ipx} + v_i(\vec{p}) b_i^+(\vec{p}) e^{ipx} \right) \ , \tag{1}$$

1. Analogously to what we did for the scalar field, let us postulate the following commutation relations

$$[a_i(\vec{p}), a_j^+(\vec{p'})] = [b_i(\vec{p}), b_j^+(\vec{p'})] = (2\pi)^3 2\omega_{\vec{p}} \delta_{ij} \, \delta^{(3)}(\vec{p} - \vec{p'}) \ , \tag{2}$$

   and zero otherwise. Compute the normal-ordered Hamiltonian. Show that it is not bounded from bellow, meaning that quantizing the Dirac field by requiring that the ladder operators obey the commutation relations (3) is inconsistent.
2. The way out is to require that the ladder operators satisfy anticommutation relations

$$\{a_i(\vec{p}), a_j^+(\vec{p'})\} = \{b_i(\vec{p}), b_j^+(\vec{p'})\} = (2\pi)^3 2\omega_{\vec{p}} \delta_{ij} \, \delta^{(3)}(\vec{p} - \vec{p'}) \ , \tag{3}$$

   and zero otherwise.
   (a) Compute the Hamiltonian of the Dirac field in terms of the ladder operators subject to (4). Is it bounded from bellow in this case ?
   (b) In the Problem Set 6 we saw that the Dirac theory is invariant under $\psi \to \psi' = e^{i\alpha}\psi$, and found that the corresponding Noether current reads $j^\mu = -\bar{\psi}\gamma^\mu\psi$. Write the charge $Q = \int d^3x j^0$ in terms of creation and annihilation operators.
   (c) Consider the state $a_i^+(\vec{p}) a_j^+(\vec{p'}) |0\rangle$. What statistics does it obey ? What is its energy and charge ? What happens if both $\vec{p} = \vec{p'}$ and $i = j$?
   (d) Find the charge of the state $b_i^+(\vec{p}) b_j^+(\vec{p'}) |0\rangle$.

2. Propagators

1. We have seen that the real scalar field admits the following mode expansion

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \left[\hat{a}(\vec{p}) e^{-ip\cdot x} + \hat{a}(\vec{p})^+ e^{ip\cdot x} \right] \ , \tag{4}$$

   with $p \cdot x = \omega_{\vec{p}} t - \vec{p} \cdot \vec{x}$.
   (a) Compute the Wightman function

$$\mathcal{D}(x - x') = \langle 0| \hat{\phi}(x) \hat{\phi}(x') |0\rangle \ . \tag{5}$$

(b) Starting from

$$\mathcal{D}_F(x,x') = \langle 0|\, T\hat{\phi}(x)\hat{\phi}(x')\, |0\rangle \ , \tag{6}$$

with T the time-ordered product

$$T\hat{\phi}(x)\hat{\phi}(x') = \theta(t-t')\hat{\phi}(x)\hat{\phi}(x') + \theta(t'-t)\hat{\phi}(x')\hat{\phi}(x) \ , \tag{7}$$

show that

$$\mathcal{D}_F(x,x') = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \left(e^{-ip\cdot(x-x')}\theta(t-t') + e^{ip\cdot(x-x')}\theta(t'-t) \right) \ . \tag{8}$$

Prove that the above can be written as

$$\mathcal{D}_F(x,x') = \lim_{\epsilon\to 0} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{ip\cdot(x-x')} \ . \tag{9}$$

Useful formula :

$$e^{-i\omega_{\vec{p}}\tau}\theta(\tau) + e^{i\omega_{\vec{p}}\tau}\theta(-\tau) = i \lim_{\epsilon\to 0} \frac{2\omega_{\vec{p}}}{2\pi} \int \mathrm{d}\omega \frac{e^{i\omega\tau}}{\omega^2 - \omega_{\vec{p}}^2 + i\epsilon} \ , \quad \epsilon > 0 \ .$$

(c) Check that for $\epsilon = 0$ the Feynman propagator satisfies the Klein-Gordon equation

$$(\Box_x + m^2)\mathcal{D}_F(x,x') = -i\delta^{(4)}(x-x') \ . \tag{10}$$

(d) Show that in position space the Feynman propagator for a massless scalar field is

$$\mathcal{D}_F(x,x') = -\frac{1}{4\pi^2} \lim_{\epsilon\to 0} \frac{1}{(x-x')^2 - i\epsilon} \ . \tag{11}$$

Hint : Start from eq. (9).
Useful formula :

$$\int_0^\infty \mathrm{d}\mu e^{i\mu a} = \lim_{\epsilon\to 0} \frac{i}{a + i\epsilon} \ .$$

2. Compute the Feynman propagator $\mathcal{S}_F(x,x')$ for the Dirac field.
Hint : You may use the fact that

$$\mathcal{S}_F(x,x') = (-i\gamma^\mu \frac{\partial}{\partial x^\mu} - m)\mathcal{D}_F(x,x') \ ,$$

with $\mathcal{D}_F(x,x')$ the Feynman propagator for the scalar field, eq. (10)