Quantum Field Theory (Quantum Electrodynamics)

Problem Set 6 25 & 27 November 2024

1. Gamma matrices in 4 spacetime dimensions, continued

The γ-matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathcal{I}_4,$$

where $\{a, b\} = [a, b]_+ = ab + ba$ is the anticommutator, and $\mathcal{I}_4 = \mathrm{diag}(1, 1, 1, 1)$ is the 4×4 identity matrix. In what follows we will not be writing $\mathcal{I}_4$ explicitly, although its presence will be tacitly assumed.

1. Prove the following identities without using any representation of the γ matrices

$$\gamma^\mu\gamma^\nu\gamma^\rho = \eta^{\mu\nu}\gamma^\rho + \eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu + i\epsilon^{\mu\nu\rho\sigma}\gamma^5\gamma_\sigma\ , \quad \text{with} \quad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3\ , \quad \epsilon_{0123} = -\epsilon^{0123} = 1.$$

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}, \quad \mathrm{tr}(\gamma_5) = \mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu) = 0,$$

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4(\eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\alpha}\eta^{\nu\beta}),$$

$$\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = -4i\epsilon^{\mu\nu\alpha\beta}.$$

$$\mathrm{tr}(\text{odd number of } \gamma\text{'s}) = 0.$$

2. The Dirac field

Consider

$$S = \int \mathrm{d}^4x \left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \right)\ , \tag{1}$$

where $\bar{\psi} = \psi^\dagger\gamma^0$, and m is the (real) mass of the field.

1. What is the dimension of ψ ?
2. Check that the action is Hermitian up to a total derivative.
   Useful formulas : $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0,\ (\gamma^0)^\dagger = \gamma^0,\ (\gamma^0)^2 = 1.$
3. Find the equations of motion for ψ and $\bar{\psi}$.
4. Check that the action is invariant under

$$\psi \to \psi' = e^{i\alpha}\psi\ ,$$

   with α a constant. Derive the corresponding Noether current. Is it conserved on the equations of motion ?
5. Find the energy-momentum tensor and verify its conservation.
6. Compute the canonical momenta π and $\bar{\pi}$.
7. Find the Hamiltonian and the three-momentum $\vec{P}$.

3. Plane-wave solutions of the Dirac equation

The Dirac equation reads

$$(i\partial\!\!\!/ - m)\psi = 0 \; , \tag{2}$$

where $\partial\!\!\!/ = \gamma^\mu \partial_\mu$. For concreteness, we will be working with the Dirac representation of the gamma matrices, see Problem Set 5.

1. Let's assume that

$$\psi = e^{-ipx} u(\vec{p}) \; ,$$

   where $u(\vec{p})$ a four-component spinor that depends only on the three-momentum $\vec{p}$.

   (a) Plug the above into (2) and write the resulting equation in matrix form. Find the condition on the four-momentum which follows from this equation.

   (b) Represent the four-component spinor in terms of two-component spinors, i.e. $u(\vec{p}) = (\xi(\vec{p}), \eta(\vec{p}))^T$ and show that the solutions to the Dirac equation read

$$u_s(\vec{p}) = \sqrt{p^0 + m} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0+m} \chi_s \end{pmatrix} \; , \quad s = 1, 2 \; ,$$

   with $\chi_1 = (1, 0)^T$ and $\chi_2 = (0, 1)^T$.

2. Using the ansatz $\psi = e^{+ipx} v(\vec{p})$, repeat the above steps and show that now the solutions read

$$v_s(\vec{p}) = -\sqrt{p^0 + m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{p^0+m} \epsilon \chi_s \\ \epsilon \chi_s \end{pmatrix} \; , \quad s = 1, 2 \; , \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

   with χ_1 and χ_2 given above.

3. Show that

$$\sum_s u_s(\vec{p}) \bar{u}_s(\vec{p}) = p\!\!\!/ + m \; , \quad \sum_s v_s(\vec{p}) \bar{v}_s(\vec{p}) = p\!\!\!/ - m \; ,$$

   where $p\!\!\!/ = \gamma^\mu p_\mu$.

4. Compute

$$\bar{u}_s(\vec{p}) u_r(\vec{p}) \; , \quad \bar{v}_s(\vec{p}) v_r(\vec{p}) \; .$$