https://www2.physik.uni-muenchen.de/lehre/vorlesungen/wise_24_25/TA1_theoretical_condensed_matter/index.html

Problem set 7

Problem 1 Fermions from strings

The lattice description of fermions is based on ladder operators and their anti-commutation relations

$$\{\hat{c}_\alpha, \hat{c}^\dagger_\beta\} = \delta_{\alpha\beta} , \quad \{\hat{c}_\alpha, \hat{c}_\beta\} = 0 , \tag{1}$$

where α, β denote single-particle states, which are used to construct the many-particle Hilbert space as discussed in the lecture, and $\{A, B\} = AB + BA$. In this exercise, we want to show that these anti-commutation relations already cause an effective, long-range interaction between the electrons on the lattice. For that purpose, we start from the lattice description of hardcore bosonic particles, which can be understood as bosons whose maximum occupation is bounded by one particle per single-particle state: $n_\alpha \leq 1$. The ladder operators for a single-particle state α are given by

$$\hat{b}_\alpha \doteq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_\alpha , \quad \hat{b}^\dagger_\alpha \doteq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_\alpha , \quad \hat{n}_\alpha \doteq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_\alpha . \tag{2}$$

(1.a) Show that the hardcore-bosonic operators obey the (non-canonical) bosonic commutation relations

$$[\hat{b}_\alpha, \hat{b}^\dagger_\beta] = \delta_{\alpha\beta} \mathrm{e}^{\mathrm{i}\pi\hat{n}_\alpha} , \quad [\hat{b}_\alpha, \hat{b}_\beta] = 0 . \tag{3}$$

(1.b) The Jordan-Wigner transformation is defined by "dressing" the hardcore bosons with so-called Jordan-Wigner strings $\hat{U}_\alpha = \mathrm{e}^{\mathrm{i}\pi \sum_{\beta<\alpha} \hat{n}_\beta}$. Show that the operators defined via

$$\hat{c}_\alpha = \hat{U}_\alpha \hat{b}_\alpha , \quad \hat{c}^\dagger_\alpha = \hat{b}^\dagger_\alpha \hat{U}^\dagger_\alpha \tag{4}$$

obey the fermionic anti-commutation relations eq. (1). How does this impliy the existence of an effective, non-local interaction between the fermions?

Problem 2 One dimension, many gaps

A simple model of a periodic potential that can be analyzed analytically to a large extent is the one-dimensional Kronig-Penney model. The periodic potential in position space has the form

$$V(x) = V_0 \sum_{n=-\infty}^{\infty} \delta(x - na) , \tag{5}$$

where a denotes the lattice constant and V_0 the coupling strength.

(2.a) Show that the dispersion is determined by the condition

$$\cos(ka) = \cos(\beta a) + \frac{v}{2\beta} \sin(\beta a) , \tag{6}$$

with $\beta = \sqrt{2mE}/\hbar$ and $v = 2mV_0/\hbar^2$.

(2.b) Show that there are energy gaps, i.e., for certain values of E no solution can be found for eq. (6).

(2.c) Examine the physical situations emerging in the limits $V_0 \to 0$ and $V_0 \to \infty$.

(2.d) Compute the density of states $\rho(E)$.