Problem set 12.

(1.a) $\vec{\pi} = \vec{p} + \frac{e}{c} A(\vec{r})$

$$[\pi_x, \pi_y] = [\hat{p}_x + \frac{e}{c} A_x(r), p_y + \frac{e}{c} A_y(r)]$$

$$= [-i\hbar\frac{\partial}{\partial x} + \frac{e}{c} A_x, -i\hbar\frac{\partial}{\partial y} + \frac{e}{c} A_y]$$

$$= -\frac{i\hbar e}{c} \left\{ [\frac{\partial}{\partial x}, A_y] - [\frac{\partial}{\partial y}, A_x] \right\}.$$

$$[\frac{\partial}{\partial x}, f] = \frac{\partial}{\partial x} f$$

$$= -\frac{i\hbar e}{c} \left\{ \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \right\} = \frac{i\hbar e}{c} \cdot B = i\frac{\hbar^2}{l^2}, \quad l^2 \equiv \frac{\hbar c}{eB}$$

(1.b) $A(\vec{r}) = By \cdot \vec{e_x}$

$$\hat{h} = \frac{1}{2m_e} \pi^2 = \frac{1}{2m_e} \{ p_y^2 + (p_x + \frac{e}{c} By)^2 \} = \frac{1}{2m_e} \{ p_y^2 + (p_x + \frac{\hbar}{l^2} y)^2 \}$$

$$= \frac{1}{2m_e} p_y^2 + \frac{1}{2m_e} \frac{\hbar^2}{l^4} (\underbrace{\frac{l^2}{\hbar} p_x}_{y_0} + y)^2 \equiv \frac{p_y^2}{2m} + \frac{1}{2} m\omega^2 y^2$$

$$\omega_c = \frac{\hbar}{m l^2}, \quad y_0 = \frac{l^2}{\hbar} p_x$$

$$a \equiv \sqrt{\frac{m\omega}{2\hbar}} (y + y_0 + \frac{i}{m\omega} p_y) = \frac{1}{\sqrt{2} l} (y + \frac{l^2}{\hbar} p_x + \frac{i l^2}{\hbar} p_y) = \frac{l}{\sqrt{2}\hbar} (\underbrace{\frac{\hbar}{l^2} y + p_x}_{\pi_x} + i p_y) = \pi_x + i\pi_y.$$

$$a^+ = \sqrt{\frac{m\omega}{2\hbar}} (y + y_0 - \frac{i}{m\omega} p_y) = \frac{l}{\sqrt{2}\hbar} (\pi_x - i\pi_y).$$

$$\hat{h} = \hbar\omega_c (\hat{n} + \frac{1}{2}).$$

(1.c)

To show: $[h_D^2, n]=0$, where $h_D = v_F(\tau_z \sigma_x \hat{\Pi}_x + \sigma_y \cdot \hat{\Pi}_y)$

$$h_D^2 = v_F^2(\sigma_x^2 \Pi_x^2 + \sigma_y^2 \Pi_y^2 + \tau_z(\sigma_x \Pi_x \sigma_y \Pi_y + \sigma_y \Pi_y \sigma_x \Pi_x))$$

Using $\Pi_y \Pi_x = \Pi_x \Pi_y - \frac{i\hbar^2}{l^2}$, $\sigma_x^2 = 1 = \sigma_y^2$

$$h_D^2 = v_F^2(I\cdot \Pi_x^2 + I\cdot \Pi_y^2 + \tau_z(\{\sigma_x, \sigma_y\}\Pi_x \Pi_y - \frac{i\hbar^2}{l^2}\overbrace{\sigma_y \sigma_x}^{-i\sigma_z}))$$

$$= v_F^2(1\cdot \Pi_x^2 + I\cdot \Pi_y^2 - \frac{\hbar^2}{l^2}\sigma_z \tau_z)$$

$$\hat{n} = a^\dagger a = \frac{l^2}{2\hbar}(\Pi_x^2 + i[\Pi_x, \Pi_y] + \Pi_y^2) = \frac{l^2}{2\hbar}(\Pi_x^2 + \Pi_y^2 - \frac{\hbar^2}{l^2}),$$

Rewrite h_D^2 in terms of n gives

$$h_D^2 = v_F^2 \frac{2\hbar^2}{l^2}(\hat{n} + \frac{1}{2} - \frac{1}{2}\tau_z \sigma_z)$$

Note that for $\hat{n} + \frac{1}{2} = (\hat{n} + \frac{1}{2}) I_{2\times 2}$.

obviously, h_D^2 contains only terms $\propto \hat{n}$ and terms $\propto 1$, therefore $[\hat{h}_D^2, \hat{n}] = 0$

And

$$\hat{h}_D^2 = \frac{2v_F^2\hbar^2}{l^2}(n + \frac{1}{2} - \frac{1}{2}\tau_z \sigma_z) = \varepsilon_D^2 W(\hat{n})$$ where $\varepsilon_0^2 = \frac{2v_F^2\hbar^2}{l^2}$, $W(\hat{n}) = (n + \frac{1}{2})\cdot I_{2\times 2} - \frac{1}{2}\tau_z \sigma_z$.

When $\tau_z = +1$ (K point), $W(\hat{n}) = \begin{pmatrix} \hat{n} & \\ & \hat{n}+1 \end{pmatrix}$; when $\tau_z = -1$ (K'), $W(\hat{n}) = \begin{pmatrix} \hat{n}+1 & \\ & \hat{n} \end{pmatrix}$

Remind that this 2x2 comes from pseudospin of sublattices A, B and $\hat{n}$ comes from spatial y DoF.

From Dirac equation (K, $\tau_z = +1$)

$$\varepsilon^2 \begin{pmatrix} \psi^A \\ \psi^B \end{pmatrix} = h_D^2 \begin{pmatrix} \psi^A \\ \psi^B \end{pmatrix} = \varepsilon_D^2 \begin{pmatrix} \hat{n} & \\ & \hat{n}+1 \end{pmatrix}\begin{pmatrix} \psi^A \\ \psi^B \end{pmatrix} \mapsto \varepsilon^2 = \varepsilon_D^2 n, \quad \begin{pmatrix} \psi^A \\ \psi^B \end{pmatrix} = e^{ikx}\begin{pmatrix} u_n^A \\ u_{n-1}^B \end{pmatrix}$$

$$\varepsilon = \pm \varepsilon_D \sqrt{n}, \quad \varepsilon_0 = \frac{\sqrt{2} v_F \hbar}{l}$$

Similarly, for $\tau_z = -1$, $\varepsilon^2 = \varepsilon_D^2 n$

$$\begin{pmatrix} \psi^A \\ \psi^B \end{pmatrix} = e^{ikx}\begin{pmatrix} u_{n-1}^A \\ u_n^B \end{pmatrix}$$

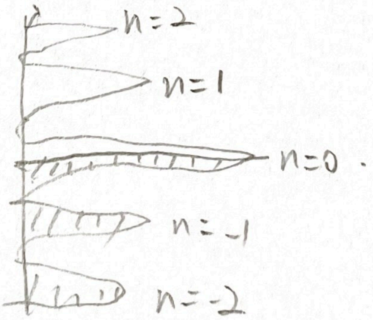
Specially, when $n=0$, since $n-1$ is unphysical, so there is only 1 band per valley per spin.

(1.d) $\sigma_{xy} = 4(n + \frac{1}{2})\frac{e^2}{h}$

2x2 spin valley (K, K')

half-filling at $n=0$

+ − → electron / hole



2. $H(R)|n(R)\rangle = \varepsilon_n(R)|n(R)\rangle$.

$$\frac{\partial}{\partial k}H(R)|n(R)\rangle + H(R)|\frac{\partial}{\partial k}n(R)\rangle = \frac{\partial}{\partial k}\varepsilon_n(R)|n(R)\rangle + \varepsilon_n(R)|\frac{\partial}{\partial k}n(R)\rangle.$$

$$\frac{\partial}{\partial k}H(R)|n(R)\rangle = \frac{\partial}{\partial k}\varepsilon_n(R)|n(R)\rangle + (\varepsilon_n(R)-H(R))|\frac{\partial}{\partial k}n(R)\rangle$$

multiply by $\langle m(R)|$ where $m \neq n$.

$$\langle m(R)|\frac{\partial}{\partial k}H(R)|n(R)\rangle = \langle m(R)|\varepsilon_n(R) - \varepsilon_m(R)|\frac{\partial}{\partial k}n(R)\rangle.$$

~~$m \leftrightarrow n$~~, $k \to j$

$$\langle n(R)|\frac{\partial}{\partial j}H(R)|m(R)\rangle = \langle\frac{\partial}{\partial j}n(R)|\varepsilon_n(R) - \varepsilon_m(R)|m(R)\rangle$$

$$\langle n(R)|\frac{\partial}{\partial j}H|m(R)\rangle\langle m(R)|\frac{\partial}{\partial k}H(R)|n(R)\rangle = \langle\frac{\partial}{\partial j}n(R)|m(R)\rangle\langle m(R)|\frac{\partial}{\partial k}n(R)\rangle(\varepsilon_n-\varepsilon_m)^2.$$

Sum over $m \neq n$ gives,

$$\sum_{m\neq n}\frac{\langle n|\frac{\partial}{\partial j}H|m\rangle\langle m|\frac{\partial}{\partial k}H|n\rangle}{(\varepsilon_n-\varepsilon_m)^2} = \langle\frac{\partial}{\partial j}n|\frac{\partial}{\partial k}n\rangle$$

$$\omega_{n,\mu\nu}(R) = \partial_\mu A^n_\nu - \partial_\nu A^n_\mu, \quad \vec{A}_n = i\langle n|\nabla_R|n\rangle = \sum_\mu i\langle n|\frac{\partial}{\partial\mu}n\rangle.$$

$$= i\{\partial_\mu\langle n|\frac{\partial}{\partial\nu}n\rangle - \partial_\nu\langle n|\frac{\partial}{\partial\mu}n\rangle = i\{\langle\frac{\partial}{\partial\mu}n|\frac{\partial}{\partial\nu}n\rangle + \langle n|\frac{\partial^2}{\partial\mu\partial\nu}n\rangle$$
$$- \langle\frac{\partial}{\partial\nu}n|\frac{\partial}{\partial\mu}n\rangle - \langle n|\frac{\partial^2}{\partial\nu\partial\mu}n\rangle\}$$

$$= i\{\langle\frac{\partial}{\partial\mu}n|\frac{\partial}{\partial\nu}n\rangle - \langle\frac{\partial}{\partial\nu}n|\frac{\partial}{\partial\mu}n\rangle\}.$$

$$= i\sum_{m\neq n}\frac{\langle n|\frac{\partial}{\partial\mu}H|m\rangle\langle m|\frac{\partial}{\partial\nu}H|n\rangle - h.c.}{(\varepsilon_n-\varepsilon_m)^2}.$$