Problem set 1.

Problem 1. Dipole Radiation

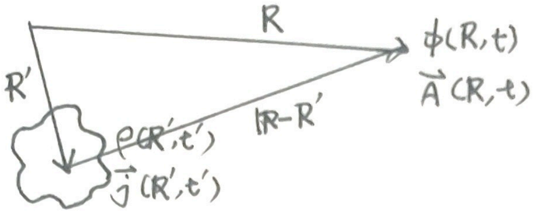
Start from Maxwell equations
$$\begin{cases} \nabla\times\vec{E}+\frac{\partial}{\partial t}\vec{B}=0 \\ \nabla\cdot\vec{B}=0 \\ \nabla\cdot\vec{E}=\rho/\varepsilon_0 \\ \nabla\times\vec{B}-\frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}=\mu_0\vec{j} \end{cases}$$

The first two equations allow us to write down scalar and vector potentials as
$$\vec{B}=\nabla\times\vec{A}\ ,\quad \vec{E}=-\nabla\phi-\frac{\partial}{\partial t}\vec{A}$$
Inserting them into source-field relations gives
$$\rho/\varepsilon_0=\nabla\cdot\vec{E}=-\nabla^2\phi-\frac{\partial}{\partial t}\nabla\cdot\vec{A}=\left\{-\nabla^2+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\}\phi-\frac{\partial}{\partial t}\left(\nabla\cdot\vec{A}+\frac{1}{c^2}\frac{\partial}{\partial t}\phi\right)$$
$$\mu_0\vec{j}=\nabla\times(\nabla\times\vec{A})+\frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}$$
Using $\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$
$$\mu_0\vec{j}=\left\{-\nabla^2+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\}\vec{A}+\nabla\left(\nabla\cdot\vec{A}+\frac{1}{c^2}\frac{\partial}{\partial t}\phi\right)$$
Using Lorentz gauge where $\nabla\cdot\vec{A}+\frac{1}{c^2}\frac{\partial}{\partial t}\phi=0$

and D'Alembert operator $\Box=-\nabla^2+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

The relation between source and four-vector potential is briefly
$$\begin{cases}\Box\phi=\rho/\varepsilon_0 \\ \Box\vec{A}=\mu_0\vec{j}\end{cases}$$
In General, Poisson equation $\Box\psi=f$ can be solved by Green's function
$$G(R,t;R',t')=\frac{1}{4\pi|R-R'|}\delta\left(c(t-t')-|R-R'|\right)$$
called retarded potential, such that
$$\psi(R,t)=\frac{1}{4\pi}\int d^3R'\,\frac{f(R',t_r)}{|R-R'|}$$
where $t_r=t-\frac{1}{c}|R-R'|$ is the retarded time

Therefore, source $\{\rho(R',t'),\vec{j}(R',t')\}$ would generate potential
$$\phi(R,t)=\frac{1}{4\pi\varepsilon_0}\int d^3R'\,\frac{\rho(R',t_r)}{|R-R'|}$$
$$\vec{A}(R,t)=\frac{\mu_0}{4\pi}\int d^3R'\,\frac{\vec{j}(R',t_r)}{|R-R'|}$$



* Far field approximation

Assume $R \gg R'$, $\frac{1}{|R-R'|} = \frac{1}{R\sqrt{1-2\frac{R'\cdot R}{R^2}+\frac{R'}{R^2}}} \approx \frac{1}{R}(1+\frac{R'\cdot R}{R^2})$

$$\vec{A}(R,t) \approx \frac{\mu_0}{4\pi}\int d^3R' \frac{\vec{j}(R',t_r)}{R}(1+\frac{R'\cdot R}{R^2}) \approx \frac{\mu_0}{4\pi R}\int d^3R' \vec{j}(R',t_r)$$

Using the continuity equation

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0, \quad \rho + \nabla\cdot\vec{p} = 0$$

Where $\vec{p}(R,t)$ is the polarization density s.t. $\frac{\partial}{\partial t}\vec{p}(R,t) = \vec{j}(R,t)$

$$\vec{A}(R,t) = \frac{\mu_0}{4\pi R}\frac{\partial}{\partial t}\int d^3R' \vec{p}(R',t_r) \equiv \frac{\mu_0}{4\pi R}\frac{\partial}{\partial t}\vec{P}(t_r)$$

Where $\vec{P}(t) = \int d^3R' \vec{p}(R',t)$ is the dipole moment.

Assume $\vec{P}(t)$ is oscillating with single frequency s.t. $\vec{P}(t) = \vec{P_0} e^{-i\omega t}$

$$\vec{A}(R,t) = \frac{\mu_0}{4\pi}(-i\omega) e^{-i\omega(t-\frac{R}{c})}\vec{P_0} = \frac{\mu_0}{4\pi}(-i\omega) e^{-i\omega t}\frac{e^{ikR}}{R}\vec{P_0}$$

where $k = \frac{\omega}{c}$, In spherical polar coordinate

$$\vec{\nabla} = \frac{\partial}{\partial R}\hat{R} + \frac{1}{R}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{R\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}$$

$$\vec{B}(R,t) = \vec{\nabla}\times\vec{A}(R,t) = \frac{\mu_0}{4\pi}(-i\omega) e^{-i\omega t}\frac{e^{ikR}}{R}(ik-\frac{1}{R})[\hat{R}\times\hat{P_0}]$$

$$\approx \frac{\mu_0}{4\pi}\frac{\omega^2}{c} e^{-i\omega t}\frac{e^{ikR}}{R}(\hat{R}\times\hat{P_0})$$

In far-field regime, $\vec{E}$ satisfies free space equation

$$\frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} = \vec{\nabla}\times\vec{B} \leftrightarrow \frac{ic^2}{\omega}\nabla\times\vec{B} = \vec{E}$$

Similarly,

$$\vec{E}(R,t) = \frac{ic^2}{\omega}\nabla\times\vec{B} \approx \frac{ic^2}{\omega}\frac{\omega^2}{c}\cdot(ik)\frac{\mu_0}{4\pi} e^{-i\omega t}\frac{e^{ikR}}{R}[\hat{R}\times[\hat{R}\times\vec{P_0}]]$$

$$= -\frac{\mu_0}{4\pi}\omega^2 e^{-i\omega t}\frac{e^{ikR}}{R}[\hat{R}\times[\hat{R}\times\vec{P_0}]]$$

$\vec{E}(R,t)$ can be rewritten as

$$\vec{E}(R,t) = \frac{\mu_0}{4\pi} \frac{e^{ikR}}{R} [\hat{R} \times [\hat{R} \times \ddot{\vec{p}}(t)]]$$

Now, shift the dipole to position $\vec{r}$ and consider the dipole to be generated by a electron under an incoming field $\vec{\varepsilon}(r,t) = \vec{E}_{in} e^{i\vec{k}\cdot\vec{r}} e^{-i\omega t}$

$\vec{p}(t)$ is related to Newton's equation by

$$\ddot{\vec{p}}(t) = -e\,\delta\ddot{\vec{r}}(t) = \frac{e^2}{m_e} \vec{\varepsilon}(\vec{r}+\delta\vec{r}, t) = \frac{e^2}{m_e} e^{i\vec{k}\cdot\vec{r}} e^{-i\omega t} \vec{E}_{in}$$

The outcoming field is

$$\vec{E}(R,t) = \frac{\mu_0 e^2}{4\pi m_e} e^{i\vec{k}\cdot\vec{r}} e^{-i\omega t} \frac{e^{ik|R-r|}}{|R-r|} [\hat{n} \times [\hat{n} \times \vec{E}_{in}]]$$

where $\hat{n} = (\vec{R}-\vec{r})/|\vec{R}-\vec{r}|$

Scaling prefactor by $4\pi\varepsilon_0$ gives $\frac{4\pi\varepsilon_0\mu_0 e^2}{4\pi m_e} = \frac{e^2}{m_e c^2}$

Power emitted

Setting $\vec{P}_0$ to z axis, one can rewrite $\vec{P}_0 = P_0 \cdot \vec{z}$

$$\vec{B} = -\frac{\omega^2 \mu_0 P_0}{4\pi c} \frac{e^{-i\omega t} \cdot e^{ikR}}{R} \sin\theta\, \hat{\phi}$$

$$\vec{E} = -\frac{\omega^2 \mu_0 P_0}{4\pi} \frac{e^{-i\omega t} e^{ikR}}{R} \sin\theta\, \hat{\theta}$$

Time averaging Poynting vector

$$\langle \vec{S} \rangle_t = \langle \tfrac{1}{2} E \times H \rangle_t = \frac{\mu_0 \omega^4 P_0^2}{32\pi^2 c} \cdot \frac{\sin^2\theta}{R^2} \cdot \hat{R}$$

$$P = \int \langle \vec{S} \rangle_t \cdot da = \frac{\mu_0 \omega^4 P_0^2}{32\pi^2 c} \int_0^{\pi} \sin^3\theta\, d\theta \int_0^{2\pi} d\varphi \cdot R^2 \cdot \frac{1}{R^2}$$

$$= \frac{\mu_0 \omega^4 P_0^2}{12\pi c}$$

Problem 2.

(2.a) $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$, $I = \int dx \, \cancel{p(x)} \, e^{-\frac{x^2}{2\sigma^2}}$

$|I|^2 = \int dx \int dy \, e^{-\frac{x^2+y^2}{2\sigma^2}}$, using polar coordinate $x = r\cos\theta$, $y = r\sin\theta$

$= \int_0^{2\pi} d\theta \int_0^{+\infty} dr \cdot r \, e^{-\frac{r^2}{2\sigma^2}} = 2\pi \int_0^{+\infty} \frac{1}{2} dr^2 \, e^{-\frac{r^2}{2\sigma^2}}$

$= 2\pi \int_0^{+\infty} du \, e^{-\frac{u}{\sigma^2}} = 2\pi \left(-\sigma^2 e^{-\frac{u}{\sigma^2}} \Big|_0^{+\infty} \right)$

$= 2\pi\sigma^2 \iff I = \sqrt{2\pi\sigma^2} \iff \int dx \, p(x) = 1$

(2.b) $\langle x^{2n+1} \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \int dx \, x^{2n+1} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} = 0$ $\quad (a = \sqrt{2}\sigma)$

$\langle x^{2n} \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \int dx \, x^{2n} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} = \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{+\infty} dx \, x^{2n} \cdot e^{-\frac{x^2}{a^2}} = \frac{2\sqrt{\pi}}{\sqrt{2\pi\sigma^2}} \frac{a^{2n+1}(2n-1)!!}{2^{n+1}}$

$= \frac{(\sqrt{2}\sigma)^{2n}}{2^n} (2n-1)!! = \sigma^{2n}(2n-1)!! = \langle x^2 \rangle^n (2n-1)!!$

(2.c) $\langle e^{i\alpha x} \rangle = \langle 1 + i\alpha x + \cdots + \frac{1}{k!} i^k \alpha^k x^k + \cdots \rangle$

$= \sum_n \frac{i^{2n} \alpha^{2n} \langle x^{2n} \rangle}{(2n)!}$

where $\langle x^{2n} \rangle = \langle x^2 \rangle^n (2n-1)!!$, using $(2n-1)!!$

$\langle e^{i\alpha x} \rangle = \sum_n \frac{(-\alpha^2)^n}{(2n)!} \frac{(2n)!}{2^n n!} \langle x^2 \rangle^n$

$= \sum_n \frac{1}{n!} \cdot \left(-\frac{\alpha^2}{2} \langle x^2 \rangle \right)^n$

$= e^{-\frac{\alpha^2}{2}\langle x^2 \rangle}$

Problem 3. Cumulant expansion

$$e^{C} = \langle e^{tA} \rangle = 1 + t\langle A \rangle + \frac{1}{2!}t^2\langle A^2 \rangle + \cdots + \frac{t^n}{n!}\langle A^n \rangle$$

$$C(t) = \log\langle e^{tA} \rangle = \sum c_n \frac{t^n}{n!}$$

$$c_n = \frac{\partial^n}{\partial t^n}\log\langle e^{tA} \rangle\Big|_{t=0}$$

Consider $\mu_k(t) = \dfrac{\langle A^k e^{tA} \rangle}{\langle e^{tA} \rangle}$, $\mu_k(0) = \langle A^k \rangle$, $\dfrac{\partial}{\partial t}\mu_k = \dfrac{\langle A^{k+1} e^{tA} \rangle}{\langle e^{tA} \rangle} - \dfrac{\langle A^{k} e^{tA} \rangle\langle A\cdot e^{tA} \rangle}{\langle e^{tA} \rangle^2}$

$$= \mu_{k+1}(t) - \mu_k(t)\mu_1(t)$$

$$c_0 = C(0) = 0$$

$$c_1 = \frac{\partial}{\partial t}\log\langle e^{tA} \rangle\Big|_{t=0} = \frac{\langle A\cdot e^{tA} \rangle}{\langle e^{tA} \rangle}\Big|_{t=0} = \mu_1(t=0) = \langle A \rangle = 0$$

$$c_2 = \frac{\partial^2}{\partial t^2}\log\langle e^{tA} \rangle\Big|_{t=0} = \frac{\partial}{\partial t}\mu_1(t)\Big|_{t=0} = \frac{\langle e^{tA} \rangle\langle A^2\cdot e^{tA} \rangle - \langle A e^{tA} \rangle^2}{\langle e^{tA} \rangle^2}\Big|_{t=0} = \mu_2(t) - \mu_1^2(t)$$

$$= \langle A^2 \rangle - \langle A \rangle^2 = \langle A^2 \rangle$$

$$c_3 = \frac{\partial}{\partial t}c_2(t)\Big|_{t=0} = \frac{\partial}{\partial t}\mu_2(t) - 2\mu_1(t)\frac{\partial}{\partial t}\mu_1(t) = \frac{\langle e^{tA} \rangle\langle A^3 e^{tA} \rangle - \langle A e^{tA} \rangle\langle A^2 e^{tA} \rangle}{\langle e^{tA} \rangle^2} - 2\mu_1(t)(\mu_2(t) - \mu_1^2(t))$$

$$= \{\mu_3(t) - \mu_1(t)\mu_2(t)\} - 2\mu_1(t)\mu_2(t) + 2\mu_1^3(t) = \mu_3(t) - 3\mu_1(t)\mu_2(t) + 2\mu_1^3(t).$$

$$= \langle A^3 \rangle$$

$$c_4 = \frac{\partial}{\partial t}c_3(t)\Big|_{t=0} = \frac{\partial}{\partial t}\mu_3(t) - 3\left\{\frac{\partial}{\partial t}\mu_1(t)\cdot\mu_2(t) + \mu_1(t)\frac{\partial}{\partial t}\mu_2(t)\right\} + 6\mu_1^2(t)\frac{\partial}{\partial t}\mu_1(t)$$

$$= \mu_4(t) - \mu_3(t)\mu_1(t) - 3\{(\mu_2 - \mu_1^2)\mu_2 + \mu_1(\mu_3 - \mu_1\mu_2)\} + 6\mu_1^2(\mu_2 - \mu_1^2).$$

$$= \mu_4(t) - 4\mu_3(t)\mu_1(t) - 3\mu_2^2(t) + 12\mu_1^2(t)\mu_2 - 6\mu_1^4(t)\Big|_{t=0}$$

$$= \langle A^4 \rangle - 3\langle A^2 \rangle^2$$