Back-of-the-Envelope Physics

Winter Term 2023/24

Sheet 10

1. Show that Maxwell's equations and the Lorentz-force equation of motion are invariant under parity P and time reversal T. In the process, determine the transformation properties of the fields $\vec{E}$, $\vec{B}$ under P and T.

2. a) Show that the convective derivative of the velocity field $\vec{v}(t, \vec{x})$ can be written as

$$D_t \vec{v} \equiv \partial_t \vec{v} + (\vec{v} \cdot \vec{\partial})\vec{v} = \partial_t \vec{v} + \vec{\Omega} \times \vec{v} + \frac{1}{2}\vec{\partial}\, \vec{v}^2 \tag{1}$$

where $\vec{\Omega} \equiv \vec{\partial} \times \vec{v}$ is the vorticity field.

b) Show that Euler's equation in a gravitational potential ϕ, $D_t \vec{v} = -(\vec{\partial} P)/\varrho - \vec{\partial}\phi$, implies Bernoulli's theorem. This states that in a stationary flow ($\partial_t \vec{v} = 0$)

$$\frac{P}{\varrho} + \frac{1}{2}\vec{v}^2 + \phi = \text{const} \tag{2}$$

along a streamline. Also, (2) holds everywhere if the stationary flow is irrotational ($\vec{\Omega} \equiv 0$).

3. Consider a cylindrical tube with inner radius a and length $l \gg a$. The z-axis is chosen to coincide with the symmetry axis of the cylinder. An incompressible fluid with dynamical viscosity μ is flowing through the tube in the z-direction. It is driven by a pressure difference ΔP between the two ends of the tube, resulting in a homogeneous pressure gradient $-\Delta P/l$ in the z-direction inside the tube. Assume that the flow has reached a steady state.

a) Using dimensional analysis, estimate the volume of the fluid per time, $\dot{V}$, flowing through the cross section of the tube.

b) Compute the radial velocity profile $v_z(r)$ inside the tube from the Navier-Stokes equation. Assume the boundary condition $v_z(a) = 0$.

c) Integrate the result of b) to obtain the exact result for $\dot{V}$.

4. Compute the speed of sound c_s for a gas with equation of state $P(\varrho) = \text{const} \cdot \varrho^\kappa$. What is κ for air, assuming adiabatic compression? Also derive the temperature dependence of c_s using the ideal gas law.