Topological Defects

Problem Sheet 12 22 January 2024

1. The Yang-Mills Instanton Equations

Let us take an $SU(2)$ pure Yang-Mills action

$$\mathcal{S} = \frac{1}{2} \int \mathrm{d}^4x \, \mathrm{Tr}(G_{\mu\nu} G^{\mu\nu}). \tag{1}$$

Note that we work now with the Euclidean metric.

1. Argue why a solution that satisfies $W_\mu \xrightarrow{r\to\infty} \frac{i}{g} U \partial_\mu U^\dagger$ with $r = \sqrt{x^\mu x_\mu}$ may give a finite action.

The winding number is given by

$$n = \frac{1}{24\pi^2} \int_{S^3_\infty} \mathrm{d}^3 S_\mu \, \mathrm{Tr} \left[\varepsilon^{\mu\nu\alpha\beta} (\partial_\nu U) U^\dagger (\partial_\alpha U) U^\dagger (\partial_\beta U) U^\dagger \right], \tag{2}$$

where U is an $SU(2)$ gauge matrix.

2. *(optional)* Show that $U^{(0)} = 1$, $U^{(1)} = (x_4 + i x_i \sigma_i)/r$, and $U^{(k)} = \left(U^{(1)}\right)^k$ describe configurations with winding number 0, 1, and k respectively.
3. Show that the winding number can be rewritten to

$$n = \frac{g^2}{8\pi^2} \int \mathrm{d}^4x \, \mathrm{Tr} \, G_{\mu\nu} \tilde{G}^{\mu\nu}, \tag{3}$$

where $\tilde{G}^{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} G^{\mu\nu}$.
Hint : Show first

$$\mathrm{Tr}\Big(G_{\mu\nu} \tilde{G}^{\mu\nu} \Big) = 2\partial_\mu \, \mathrm{Tr} \left(W_\nu \partial_\alpha W_\beta - \frac{4i}{3} g W_\nu W_\alpha W_\beta \right) \varepsilon^{\mu\nu\alpha\beta}. \tag{4}$$

4. Derive the Bogomolny bound

$$S \geq \pm \frac{1}{2} \int \mathrm{d}^4x \partial_\mu \, \mathrm{Tr} \left(W_\nu G_{\alpha\beta} + \frac{2i}{3} g W_\nu W_\alpha W_\beta \right) \varepsilon^{\mu\nu\alpha\beta} = \pm \frac{4\pi^2}{g^2} n. \tag{5}$$

5. Show that the instanton equations

$$G_{\mu\nu} = \tilde{G}_{\mu\nu} \qquad \text{for } n > 0, \tag{6}$$
$$G_{\mu\nu} = -\tilde{G}_{\mu\nu} \qquad \text{for } n < 0, \tag{7}$$

need to be satisfied if we want the inequality to be equality.

2. The Yang-Mills Instanton Solution

In this problem, we want to find a solution with a winding number 1 that satisfies the self-duality equation $G_{\mu\nu} = \tilde{G}_{\mu\nu}$.

1. Show that you can rewrite the solution that we found in problem 1 in the long-range limit for winding number 1 to

$$W_\mu^a \xrightarrow{r\to\infty} \frac{2}{g}\eta_{a\mu\nu}\frac{x^\nu}{r^2}, \tag{8}$$

where $\eta_{a\mu\nu}$ are the 't Hooft symbols that are given by

$$\eta_{a\mu\nu} = \begin{cases} \varepsilon_{a\mu\nu} & \mu, \nu = 1, 2, 3 \\ -\delta_{a\nu} & \mu = 4 \\ \delta_{a\mu} & \nu = 4 \end{cases} \tag{9}$$

The 't Hooft symbols satisfy the following two identities

$$\varepsilon_{abc}\eta_{b\alpha\beta}\eta_{c\gamma\delta} = \delta_{\alpha\gamma}\eta_{a\beta\delta} - \delta_{\alpha\delta}\eta_{a\beta\gamma} - \delta_{\beta\gamma}\eta_{a\alpha\delta} + \delta_{\beta\delta}\eta_{a\alpha\gamma}, \tag{10}$$

$$\varepsilon_{\alpha\beta\gamma\delta}\eta_{a\mu\delta} = \delta_{\mu\alpha}\eta_{a\beta\gamma} - \delta_{\mu\beta}\eta_{a\alpha\gamma} + \delta_{\gamma\mu}\eta_{a\alpha\beta}. \tag{11}$$

Since you don't learn anything from the proof, you don't have to check these identities.
So far we know the solution in the long-range limit. But as we already did for the vortex/string solution and the magnetic monopole solution we can multiply to the long-range limit a profile function to get a solution that is valid everywhere

$$W_\mu^a = \frac{2}{g}\eta_{a\mu\nu}\frac{x^\nu}{r^2}f(r) \tag{12}$$

2. Insert this solution into the self-duality equation and find

$$2f(f-1) + r\frac{\mathrm{d}f}{\mathrm{d}r} = 0. \tag{13}$$

3. Show that the solution gives you

$$W_\mu^a = \frac{2}{g}\eta_{a\mu\nu}\frac{x^\nu}{r^2+\rho^2} \tag{14}$$

with ρ being a constant.

3. The Strong CP Problem

Remember that in the path integral formulation, a transition amplitude is given by

$$\langle\phi_F| e^{iHT} |\phi_I\rangle = \int \mathrm{D}\phi\, e^{iS}. \tag{15}$$

In Euclidean space-time, this can be written as

$$\langle\phi_F| e^{-HT_E} |\phi_I\rangle = \int \mathrm{D}\phi\, e^{-S_E}. \tag{16}$$

1. Take a $0+1$ dimensional scalar field theory with potential $V = \lambda(\phi^2 - v^2)^2$ and argue why $\langle -v| e^{-HT_E} |v\rangle$ is not vanishing.

Now let us analyze a Yang-Mills theory in $3+1$ dimensions.

2. Argue why $\langle n + \Delta n| e^{-iHT} |n\rangle$ is not vanishing, where n describes the winding number defined in problem 1.
3. Explain why $|n\rangle$ is not an eigenstate of all gauge transformation.
4. Since $|n\rangle$ is not gauge invariant, it does not describe the true vacuum. Show that the so-called θ-vacuum

$$|\theta\rangle = \sum_{n=-\infty}^{+\infty} e^{in\theta} |n\rangle \tag{17}$$

   describes a gauge invariant vacuum.
5. Show that the vacuum-to-vacuum transition amplitude can be written as

$$\langle\theta'| e^{-iHT} |\theta\rangle = 2\pi\delta(\theta - \theta') \int \mathrm{D}W e^{iS[W]-i\Delta S[W]}. \tag{18}$$

   Determine $\Delta S[W]$.
6. Show that this new extra term is not CP invariant.