Problem 1: (Here we follow Weinberg, Guth - Nonexistence of spherically symmetric monopoles with multiple magnetic charge)

(1)

infinitesimal rotation matrix $\omega_{ij} = \omega_k \varepsilon_{kij}$ rotation axis

- Rotation: $\delta x_i = \omega_{ij} x_j$

$$\delta \vec{\Phi} = \vec{\Phi}(x_i + \omega_k \varepsilon_{kij} x_j) - \vec{\Phi}(x_i)$$
$$= \vec{\Phi}(x_i) + \omega_k \varepsilon_{kij} x_j \partial_i \vec{\Phi}(x_i) - \vec{\Phi}(x_i)$$
$$= -\omega_k \delta_k \vec{\Phi}$$

minus is convention

$$\rightarrow \delta_i \vec{\Phi} = \varepsilon_{ijk} x_j \partial_k \vec{\Phi}$$

$$\delta \vec{W}_i = (\delta_{ij} + \omega_k \varepsilon_{kij}) \vec{W}_j(x_i + \omega_k \varepsilon_{kij} x_j) - \vec{W}_i(x_i)$$
$$\approx \vec{W}_i(x_i + \omega_k \varepsilon_{kij} x_j) + \omega_k \varepsilon_{kij} \vec{W}_j(x_i) - \vec{W}_i(x_i)$$
$$\approx \vec{W}_i(x_i) + \omega_k \varepsilon_{kmj} x_j \partial_m \vec{W}_i(x_i) + \omega_k \varepsilon_{kij} \vec{W}_j(x_i) - \vec{W}_i(x_i)$$
$$= -\omega_j \delta_j \vec{W}_i$$

$$\rightarrow \delta_i \vec{W}_j = \varepsilon_{imn} x_m \partial_n \vec{W}_j + \varepsilon_{ijk} \vec{W}_k$$

- The variations $\delta_i \vec{\Phi}$ and $\delta_i \vec{W}_j$ as they are written above will never vanish. But we are always allowed to make additionally a gauge transformation

$$\delta_i \vec{\Phi} = \varepsilon_{ijk} x_j \partial_k \vec{\Phi} + \vec{\Lambda}_i \times \vec{\Phi}$$

$$\delta_i \vec{\Phi} = \varepsilon_{ijk}\, x_j\, \partial_k \vec{\Phi} + \vec{\Lambda}_i \times \vec{\Phi}$$

$$\delta_i \vec{W}_j = \varepsilon_{imn}\, x_m\, \partial_n \vec{W}_j + \varepsilon_{ijk} \vec{W}_k + \vec{\Lambda}_i \times \vec{W}_j - \frac{1}{g} \partial_j \vec{\Lambda}_i$$

If there is a $\vec{\Lambda}_i$, such that $\delta_i \vec{\Phi}, \delta_i \vec{W}_j = 0$, then there exists a spherically symmetric solution.

(2)

- We fix direction of $\vec{\Phi}$ by

$$\vec{\Phi} = h(\vec{x})\, \vec{w}$$

$$\vec{w} = \begin{pmatrix} \cos(n\varphi) \sin\Theta \\ \sin(n\varphi) \sin\Theta \\ \cos\Theta \end{pmatrix}$$

because we know that this configuration gives a multiple magnetic charge.

This fixes the gauge partially.

We can still do gauge transformations with $\vec{\Lambda}$ parallel to $\vec{\Phi}$.

- $$x_i \vec{W}_i' \cdot \vec{w} = x_i \vec{W}_i \cdot \vec{w} + x_i \underbrace{(\alpha_j \vec{\Lambda}_j \times \vec{W}_i) \cdot \vec{w}}_{=0,\ \text{because } \vec{\Lambda}_j \parallel \vec{w}} - x_i \frac{1}{g} \alpha_j \partial_i \Lambda_j \overset{!}{=} 0$$

$$\rightarrow x_i \vec{W}_i \cdot \vec{w} = \frac{1}{g} x_i\, \alpha_j\, \partial_i \Lambda_j$$

This equation can be satisfied and thus we can demand $x_i \vec{W}_i \cdot \vec{w} = 0$

(3)

- We want to have a rotationally symmetric configuration, which means that the gauge configuration, which means that the gauge conditions also have to be rotationally symmetric. This constraints the possible gauge freedom further.

- $\vec{\Phi} = h(r)\vec{w}$

$\neq 0$, because this is not a pure $\underbrace{\text{gauge}}\ \underbrace{\text{trafo.}}$

$$0 \overset{!}{=} \delta_i \vec{\Phi} = \varepsilon_{ijk} x_j \partial_k \vec{\Phi} + \vec{\Lambda}_i \times \vec{\Phi}$$

$$= \varepsilon_{ijk} \cancel{x_j \frac{x_k}{r}} h'(r)\vec{w} + \varepsilon_{ijk} x_j h(r)(\partial_k \vec{w}) + \vec{\Lambda}_i \times \vec{w} \quad h(r) | \vec{w}\times$$

$$= \varepsilon_{ijk} x_j h(r)\, \vec{w} \times (\partial_k \vec{w}) + \vec{w} \times (\vec{\Lambda}_i \times \vec{w})\, h(r)$$

$$= -\varepsilon_{ijk} x_j h(r)(\partial_k \vec{w}) \times \vec{w} + \vec{\Lambda}_i h(r) - \underbrace{(\vec{w}\cdot\vec{\Lambda}_i)}_{=:\ f_i(x_i)} \vec{w}\ h(r)$$

$$\rightarrow \vec{\Lambda}_i = \varepsilon_{ijk} x_j (\partial_k \vec{w}) \times \vec{w} + f_i(x_i)\vec{w} \qquad (1)$$

- $x_i \vec{W}_i \cdot \vec{w} = 0$

$$0 \overset{!}{=} x_j \delta_i \vec{W}_j \cdot \vec{w}$$

$$\overset{(1)}{=} x_j \varepsilon_{imn} x_m (\partial_n \vec{W}_j)\cdot \vec{w} + x_j \varepsilon_{ijk} \vec{W}_k \cdot \vec{w}$$
$$+ x_j \vec{w} \cdot \left[(\varepsilon_{imk} x_m (\partial_k \vec{w}) \times \vec{w} + f_i \vec{w}) \times \vec{W}_j\right]$$
$$- \frac{1}{g} x_j \vec{w} \cdot \partial_j \left(\varepsilon_{imk} x_m (\partial_k \vec{w}) \times \vec{w} + f_i \vec{w}\right) \qquad (2)$$

$$= \partial_n \left(\varepsilon_{imn} x_m \underbrace{x_j \vec{W}_j \cdot \vec{w}}_{=0}\right) - \varepsilon_{imn} \cancel{\delta_{nj}} x_m \vec{W}_j \cdot \vec{w}$$

$$- \varepsilon_{imn} \delta_{mn} \underbrace{x_j \vec{W}_j \cdot \vec{w}}_{=0} - \varepsilon_{imn} x_j \cancel{x_m \vec{W}_j} \cdot \partial_n \vec{w} + x_j \cancel{\varepsilon_{ijk} \vec{W}_k} \cdot \vec{w}$$

$$+ x_j \vec{W}_j \cdot \Big[\underbrace{\vec{w} \times (\varepsilon_{imk} x_m (\partial_k \vec{w}) \times \vec{w}}_{= \varepsilon_{imk} \cancel{x_m} \partial_k \vec{w} - 0} + f_i \cancel{\vec{w}})\Big]$$

$$= \varepsilon_{imk} \cancel{x_m} \partial_k \vec{w} - 0$$

$$- \frac{1}{g} x_j \vec{w} \cdot \Big[\varepsilon_{imk} \delta_{mj} (\partial_k \cancel{\vec{w}}) \times \vec{w} + \varepsilon_{imk} x_m (\partial_k \partial_m \cancel{\vec{w}}) \times \vec{w}$$

$$+ \varepsilon_{imk} x_m (\partial_k \vec{w}) \times (\partial_j \vec{w}) + (\partial_j f_i) \vec{w} + f_i \cancel{\partial_j \vec{w}} \Big]$$

$$= - \frac{1}{g} \varepsilon_{imk} x_m \Big[(\partial_k \vec{w}) \times (\underbrace{x_j \partial_j \vec{w}}_{= r \cdot \underbrace{\partial_r \vec{w}}_{=0}}) \Big] \cdot \vec{w} - \frac{1}{g} x_j \partial_j f_i$$

$$= - \frac{r}{g} \partial_r f_i$$

$\ulcorner \vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \dots\dots \lrcorner$

$$\rightarrow \quad \frac{\partial f_i}{\partial r} = 0$$

(4)

- $\vec{W}_i = \vec{A}_i + \vec{W}_i^\infty$ with $\vec{W}_i \xrightarrow{r \to \infty} \vec{W}_i^\infty$

rotational invariance requires $\delta_i \vec{W}_j^\infty = 0$

- $0 = \delta_i \vec{W}_j^\infty \cdot \vec{w}$

same calculation as above $\nearrow(2)$
$$= \varepsilon_{imn} x_m (\partial_n \vec{W}_j^\infty) \cdot \vec{w} - \varepsilon_{ijk} \vec{W}_k^\infty \cdot \vec{w} - \frac{1}{g} \partial_j f_i$$

(only first line remains and term with $\partial_i f_j$)

With $\vec{W}_i^\infty = \frac{1}{g} \partial_i \vec{w} \times \vec{w}$ (see problem sheet 8.3)

$$0 = \frac{1}{g} \varepsilon_{imn} x_m \Big((\partial_n \cancel{\partial_j \vec{w}}) \times \vec{w} + (\partial_j \vec{w}) \times (\partial_n \vec{w}) \Big) \cdot \vec{w}$$

$$- \frac{1}{g} \varepsilon_{ijk} \Big((\partial_k \vec{w}) \cancel{\times \vec{w}} \Big) \cdot \vec{w} - \frac{1}{g} \partial_j f_i$$

$$\rightarrow \partial_j f_i = \varepsilon_{imn} x_m \Big((\partial_j \vec{w}) \times (\partial_n \vec{w}) \Big) \cdot \vec{w}$$

problem sheet 8.2

problem sheet 8.2

$$\stackrel{\text{problem sheet 8.2}}{=} \varepsilon_{imn}\, x_m \left(n\, \varepsilon_{jnk} \frac{x_k}{r^3} \right)$$

$$= n \frac{x_m x_k}{r^3} \left(\delta_{ki} \delta_{mj} - \delta_{km} \delta_{ij} \right)$$

$$= - \frac{n}{r} \left(\delta_{ij} - \frac{x_i x_j}{r^2} \right) \qquad (3)$$

$\rightarrow$ $\partial_j f_i$ is symmetric in i and j.

- $\varepsilon_{kji}\, \partial_j f_i = (\vec{\nabla} \times \vec{f})_k = 0$

  $\rightarrow$ We can write f_i as the gradient of some scalar function: $f_i = \partial_i g$

(5)

- Inserting $f_i = \partial_i g$ in (3):

  $$\partial_i \partial_i g = - \frac{n}{r} (3 - 1)$$

  $$\underbrace{\Delta g}_{\text{length units } -2} = - \underbrace{\frac{2n}{r}}_{\text{length units } -1}$$

- g has length units 1

  $\rightarrow$ $g(x) = ar + b_i x_i$

  From the dimensional analysis we could also add other terms like $c_{ij} \frac{x_i x_j}{r}$, but they are not consistent with $\frac{\partial f_i}{\partial r} = 0$.

- $\Delta g = \partial_i \left(a \frac{x_i}{r} + \overset{\text{constant}}{b_i} \right)$

- $\Delta g = \partial_i \left(a \frac{x_i}{r} + b_i \right)$

$$= \delta_{ii} \frac{a}{r} - a \frac{x_i x_i}{r^3} = \frac{2a}{r}$$

$\rightarrow a = -n$

$\rightarrow g = -nr + b_i x_i$

(6)

- rotational invariance $\quad \delta_i \vec{W}_j = 0$

$$\delta_i \vec{W}_j = \underbrace{\delta_i \vec{W}_j^\infty}_{=0 \text{ for }} + \delta_i \vec{A}_j \overset{!}{=} 0$$

$\vec{\Lambda}_i = \varepsilon_{ijk} x_j (\partial_k \vec{w}) \times \vec{w} + \partial_i g \, \vec{w}$

(note that $-\frac{1}{g} \partial_j \vec{\Lambda}_i$ is included in $\delta_i \vec{W}_j^\infty$)

$$\rightarrow 0 = \varepsilon_{imn} x_m \partial_n \vec{A}_j + \varepsilon_{ijk} \vec{A}_k + \vec{\Lambda}_i \times \vec{A}_j \qquad \Big| \cdot \frac{x_i}{r}$$

$$0 = \frac{x_i}{r} \varepsilon_{ijk} \vec{A}_k + \frac{x_i}{r} (\partial_i g) \; \vec{w} \times \vec{A}_j$$

- Inserting g gives

$$\frac{x_i}{r} \varepsilon_{ijk} \vec{A}_k = \left(n - \frac{x_i b_i}{r} \right) \vec{w} \times \vec{A}_j \qquad (4)$$

(7)

- $(4) \cdot \vec{w} \quad \rightarrow \quad \frac{x_i}{r} \varepsilon_{ijk} \vec{A}_k \cdot \vec{w} = 0$

and from part 2 we know $\quad x_i \vec{A}_i \cdot \vec{w} = 0$

$\rightarrow \vec{A}_i \cdot \vec{w} = 0$

($\vec{x} \times \vec{v} = 0$ and $\vec{x} \cdot \vec{v} = 0$ implies $\vec{v} = 0$)

- Squaring (4):

$$\left(\frac{x_i}{r} \varepsilon_{ijk} \vec{A}_k \right) \left(\frac{x_m}{r} \varepsilon_{mjn} \vec{A}_n \right) = \left(n - \frac{x_i b_i}{r} \right) \left(n - \frac{x_k b_k}{r} \right) (\vec{w} \times \vec{A}_j) \cdot (\vec{w} \times \vec{A}_j)$$

$$\left(\frac{x_i}{r}\varepsilon_{ijk}\vec{A}_k\right)\left(\frac{x_m}{r}\varepsilon_{mjn}\vec{A}_n\right)=\left(n-\frac{x_i b_i}{r}\right)\left(n-\frac{x_k b_k}{r}\right)(\vec{w}\times\vec{A}_j)\cdot(\vec{w}\times\vec{A}_j)$$

$$\frac{x_i x_i}{r^2}\vec{A}_k\cdot\vec{A}_k-\left(\frac{x_i}{r}\vec{A}_i\right)^2=\left(n-\frac{x_i b_i}{r}\right)^2\vec{A}_j\cdot\underbrace{((\vec{w}\times\vec{A}_j)\times\vec{w})}_{=-\vec{w}\underbrace{(\vec{w}\cdot\vec{A}_j)}_{=0}+\vec{A}_j}$$

$$-\left(\frac{x_i}{r}\vec{A}_i\right)^2=\left[\left(n-\frac{x_i b_i}{r}\right)^2-1\right]\vec{A}_i\cdot\vec{A}_i$$

(8)

- $\left(\frac{x_i}{r}\vec{A}_i\right)^2\geq 0$ and $(\vec{A}_i\cdot\vec{A}_i)\geq 0$

  $\rightarrow\ 1-\left(n-\frac{x_i b_i}{r}\right)^2\overset{!}{\geq}0$

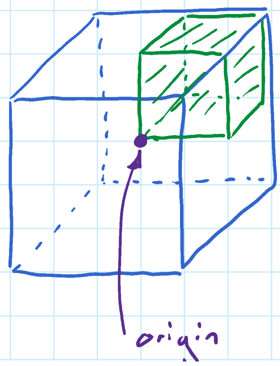
  For $n=0,\pm 1$ this is fine for example if we choose $b_i=0$.

- For $|n|\geq 2$:

  a) $\frac{x_i b_i}{r}>0$ $\rightarrow$ There may be a non-trivial solution

  b) $\frac{x_i b_i}{r}<0$ $\rightarrow$ only $\vec{A}_i=0$ is a solution

  This means that there is at least one eighth of the whole space in which $\vec{A}_i=0$ and $\vec{W}_i=\vec{W}_i^\infty=\frac{1}{g}(\partial_i\vec{w})\times\vec{w}$



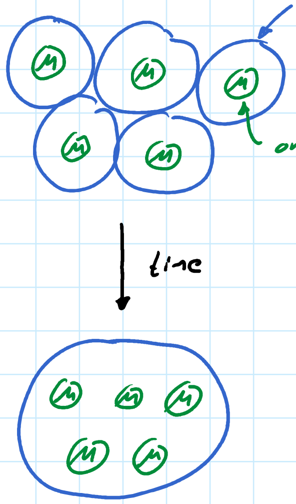
- $\bar{W}_i^\infty \sim \frac{1}{r}$ $\rightarrow$ energy density $\sim \frac{1}{r^4}$

  $\rightarrow$ infinite contribution to energy at the origin

$\Rightarrow$ there is no spherical symmetric solution with finite energy for $|n| \geq 2$.

Problem 2:

(1)



causally disconnected regions of radius $r_H \sim \frac{M_{Pl}}{T_{GUT}^2}$

one monopole per c.d. region

- Assumptions: - monopoles don't annihilate during expansion
  - photon number constant

  $\rightarrow$ number of monopoles per photon is constant

- $$\left.\begin{aligned} n_M^{GUT} &\sim \frac{1}{r_H^3} \sim \frac{T_{GUT}^6}{M_{Pl}^3} \\ n_\gamma^{GUT} &\sim T_{GUT}^3 \end{aligned}\right\} \quad \frac{n_M^{GUT}}{n_\gamma^{GUT}} \sim \left(\frac{T_{GUT}}{M_{Pl}}\right)^3$$

- today $\frac{n_M^0}{n_\gamma^0} = \frac{n_M^{GUT}}{n_\gamma^{GUT}} \sim \left(\frac{T_{GUT}}{M_{Pl}}\right)^3$

$(1eV \sim 10^4 K)$

$\rightarrow \varepsilon_M \sim m_M n_M \sim m_M \left(\frac{T_{GUT}}{M_{Pl}}\right)^3 \cdot T_0^3$ $\quad (T_0 \approx 2{,}73 K \sim 10^{-13} GeV)$

$\sim 10^{17} GeV \cdot \left(\frac{10^{16}}{10^{19}}\right)^3 \cdot (10^{-13} GeV)^3$

$\sim 10^{-31} GeV^4 \cdot \frac{1}{\hbar^3 c^5}$

$\sim 10^{-31} (10^9 \cdot 10^{-19} \; J)^4 \cdot \frac{1}{(10^{-34} Js)^3 (10^8 \frac{m}{s})^5}$

$\sim 10^{-9} \frac{kg}{m^3}$

$\sim 10^{-12} \frac{g}{cm^3} \gg 10^{-31} \frac{g}{cm^3}$

(density of universe)

$\rightarrow$ magnetic monopoles would dominate the universe, but so far we never observed one. Furthermore, the universe would be matter-dominated which is also inconsistent with our observations.

(2)

- $V = \frac{\lambda}{8} (\Phi^a \Phi^a - v^2)^2 \, \Phi^a \Phi^a$

Let us fix for the moment $\Phi = (\varphi, 0, 0)$

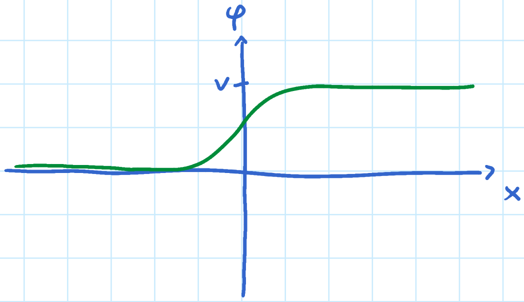
$V = \frac{\lambda}{8} (\varphi^2 - v^2)^2 \varphi^2$

$\rightarrow$ The DW solution we already found on sheet 3:

φ

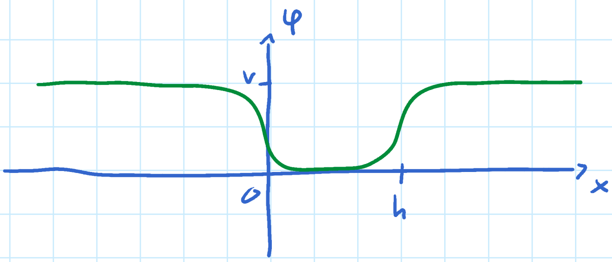
sheet 3:

$$\varphi_{DW} = \frac{\pm v}{\sqrt{1 - e^{\pm m_h x}}}$$

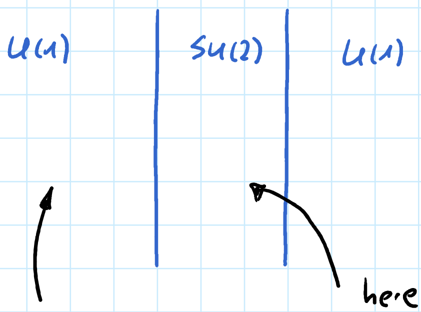


(3)

- $$\varphi_{VL} = \frac{v}{\sqrt{1-e^{-m_h x}}} + \frac{v}{\sqrt{1-e^{m_h (x-h)}}}$$ (see sheet 3)



(4)



here the monopole can exist, because $SU(2) \to U(1)$

here the monopole cannot exist, because $SU(2)$ is not broken

- If a monopole in the $U(1)$ phase encounters a vacuum layer in which the $SU(2)$ symmetry is restored, it will disappear, because it cannot exist in the $SU(2)$ vacuum.

(5)

- magnetic charge spreads inside vacuum layer

$\rightarrow$ emission of electromagnetic radiation

→ emission of electromagnetic radiation

- mass of the monopole disappears

→ emission of gravitational radiation

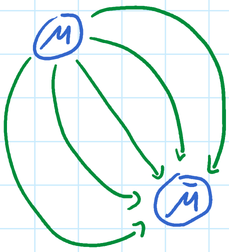
(6)

- $V = a \cdot (\mathrm{Tr}(\Phi^\dagger\Phi) - v_\Phi^2)^2 + b\,(\Psi^\dagger\Psi - v_\Psi^2)^2 + c\ \Psi^\dagger\Phi\Psi$

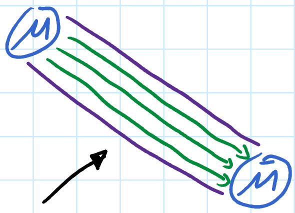
with $v_\Phi > v_\Psi$ to have the right breaking hierarchy.

(7)

- monopole-antimonopole:



monopole-antimonopole connected by string:



higher density of magnetic field lines

→ stronger attraction