10 Problem Sheet

Freitag, 15. Dezember 2023 08:25

Problem 1:

Φ_0 minimum, G gauge group

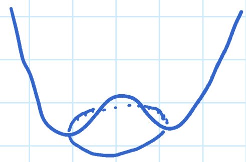
orbit of Φ_0: $\mathcal{O}_{\Phi_0} = \{\Phi = g\Phi_0 : g \in G\}$

vacuum manifold: $$\mathcal{M} = \bigcup_{\Phi_0 \text{ minimizes } V} \mathcal{O}_{\Phi_0}$$

(1)

- One minimum of this theory is $\Phi = v$

  $\mathcal{O}_{\Phi=v} = \{ g \cdot v : g \in U(1)\} = \{ e^{i\alpha} v : \alpha \in [0, 2\pi]\}$



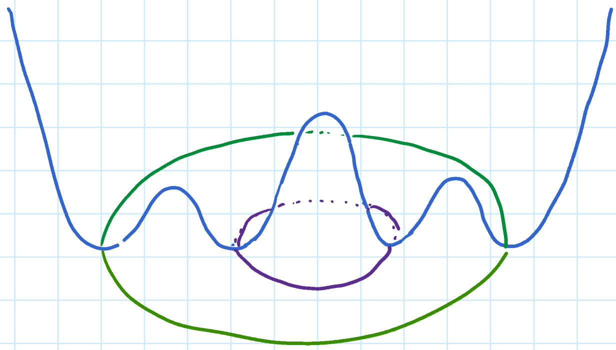
- $\mathcal{M} = \mathcal{O}_{\Phi=v} = \{ v \cdot e^{i\alpha} : \alpha \in [0, 2\pi]\}$

(2)

- There are two orbits:

  $\mathcal{O}_{\Phi=v_1} = \{ v_1 e^{i\alpha} : \alpha \in [0, 2\pi]\}$

  $\mathcal{O}_{\Phi=v_2} = \{ v_2 e^{i\alpha} : \alpha \in [0, 2\pi]\}$



- $\mathcal{M} = \mathcal{O}_{\Phi=v_1} \cup \mathcal{O}_{\Phi=v_2}$

  $= \{ \rho e^{i\alpha} : \alpha \in [0, 2\pi], \rho \in \{v_1, v_2\}\}$

Comment: Notice that $\mathcal{O}_{\Phi=v_1} = \mathcal{O}_{\Phi=iv_1} = \mathcal{O}_{\Phi=\frac{1}{\sqrt{2}}(v_1+iv_1)} = \ldots$

stabilizer subgroup: $H_{\Phi_0} = \{ h \in G, \Phi \in \mathcal{O}_{\Phi_0} : \Phi = h\Phi\}$

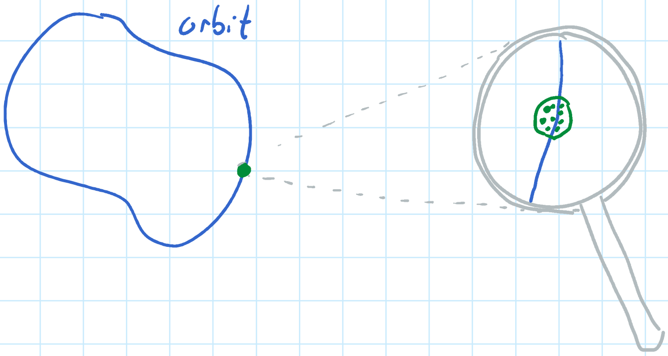
coset: $gH_{\Phi_0} = \{ gh : h \in H_{\Phi_0} \}$

coset: $$gH_{\Phi_0} = \{gh : h \in H_{\Phi_0}\}$$

coset space: $$G/H_{\Phi_0} = \{gH_{\Phi_0} : g \in G\}$$

$$G/H = \bigsqcup_{\text{inequiv. orbits}} G/H_{\Phi_0}$$

there is a one-to-one correspondence between G/H_{Φ_0} and $\mathcal{O}_{\Phi_0}$



one coset collects all transformations in G that keep Φ_0 fixed.

disjoint union: every element keeps the information from which set it is coming:

$$\{1,2\} \sqcup \{1,2,3\} = \{(1,1),(2,1),(1,2),(2,2),(3,2)\}$$

there is a one-to-one correspondence between G/H and $\mathcal{M}$

(3)

- V_1: $H_{\Phi_0} = \{\mathbb{1}\}$

$$gH_{\Phi_0} = \{e^{i\alpha} : \alpha \in [0, 2\pi]\}$$

$$G/H_{\Phi_0} = \{\{e^{i\alpha} : \alpha \in [0,2\pi]\}\} = G/H$$

there is one coset that corresponds to one orbit

- V_2: $H_{v_1} = \{\mathbb{1}\}$

$$H_{v_2} = \{\mathbb{1}\}$$

$$gH_{v_1} = gH_{v_2} = \{g\}$$

$$g^{H_{v_1}} = g^{H_{v_2}} = \{g\}$$
$$G/H_{v_1} = G/H_{v_2} = \{\{e^{i\alpha}\} : \alpha \in [0,2\pi]\}$$

$$G/H = \{(\{\{e^{i\alpha}\} : \alpha \in [0,2\pi]\}, 1), (\{\{e^{i\alpha}\} : \alpha \in [0,2\pi]\}, 2)\}$$

In G/H are two cosets that correspond to the two orbits.

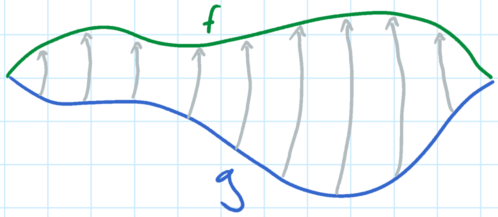
- In the second example it becomes clear why we need the disjoint union. Without it we would have
$$G/H_{v_1} \cup G/H_{v_2} = \{\{e^{i\alpha}\} : \alpha \in [0,2\pi]\}$$
Therefore, this set would have only one element instead of two and thus we wouldn't have a one-to-one correspondence anymore.

(4)

- as we already know this potential leads to the breaking $SU(2) \longrightarrow U(1)$.
One minimum is for example $\Phi = v T_3$
$$\mathcal{O}_{vT_3} = \{e^{i\alpha^a T^a} \, v T_3 \, e^{-i\alpha^a T^a} : \alpha^a \in [0,2\pi]\} = \mathcal{M}$$

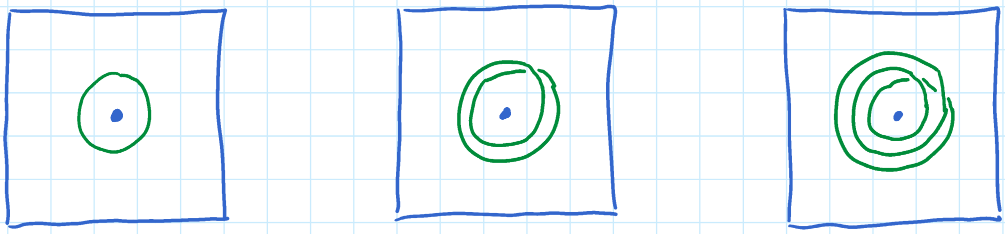
- $H_{vT_3} = \{e^{i\alpha T^3} : \alpha \in [0,2\pi]\}$
$$g^{H_{vT_3}} = \{g e^{i\alpha T^3} : \alpha \in [0,2\pi]\}$$
$$G/H_{vT_3} = \{\{e^{i\beta^a T^a} e^{i\alpha T^3} : \alpha \in [0,2\pi]\} : \beta^1, \beta^2 \in [0,2\pi], \beta^3 = 0\} = G/H$$
$$\simeq S_2$$

two mappings f and g are homotopic if they can be continously deformed into each other:



homotopy class: set of homotopic mappings

(5)

- 

$f_1(\alpha) = e^{i\alpha}$ $\quad f_2(\alpha) = e^{i2\alpha}$ $\quad f_3(\alpha) = e^{i3\alpha}$

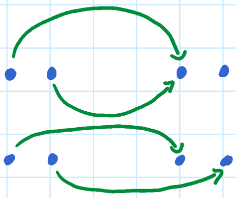
→ there are infinitely many non-homotopic mappings:

$f_n(\alpha) = e^{in\alpha}$ with $n \in \mathbb{Z}$

homotopy group $\pi_n(G)$: group of all homotopy classes with mappings from S_n to G.

(6)

- $\pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$

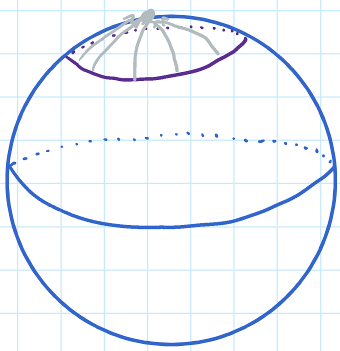


two possible non-homotopic maps

- $\pi_1(S_1) = \mathbb{Z}$ (see part (5), $S_1 \simeq \mathbb{R}^2 \setminus (0,0)$)

(7)

- $\pi_1(S_2) = 1$ (sometimes also denoted by 0)



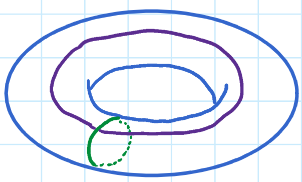
All these mappings can be contracted to one single point. All mappings are homotopic to a mapping that maps S_1 to a single point.

two useful rules:

$$\pi_n(G_1 \times G_2) = \pi_1(G_1) \times \pi_2(G_2)$$

$$\pi_2(G/H) = \pi_1(H)$$

(8)



there are two possible ways to map.

$$\pi_1(T_2) = \mathbb{Z} \times \mathbb{Z}$$

(9)

- $G = SU(2), \quad H = U(1)$

$$\pi_2(SU(2)/U(1)) = \pi_1(U(1)) = \mathbb{Z}$$

(10)

- GUT: $SU(5) \to SU(3) \times SU(2) \times U(1) = G_{SM}$

$$\pi_2(SU(5)/G_{SM}) = \pi_1(SU(3) \times SU(2) \times U(1))$$

$$= \underbrace{\pi_1(SU(3))}_{=1} \times \underbrace{\pi_1(SU(2))}_{=1} \times \underbrace{\pi_1(U(1))}_{=\mathbb{Z}}$$

$$= \mathbb{Z}$$

→ second homotopy group of $SU(5)$ is nontrivial and thus magnetic monopoles may appear in the phase transition

$$SU(5) \to SU(3) \times SU(2) \times U(1)$$

Problem 2:

(1)

- The domain wall should separate

$$\langle \Phi \rangle = +v \quad \text{and} \quad \langle \Phi \rangle = -v$$

i.e. it should be in the real space.

Let us set $\mathrm{Im}\,\Phi = 0$ for the moment.

- $\mathcal{L} = \partial_\mu \Phi_r \partial^\mu \Phi_r - \lambda (\Phi_r^2 - v^2)^2$

$$\to \quad 2\,\partial_\mu \partial^\mu \Phi_r + \frac{\partial V}{\partial \Phi_r} = 0$$

static equation: $\quad 2\,\partial_x^2 \Phi_r = \dfrac{\partial V}{\partial \Phi_r} \qquad \Big| \cdot \dfrac{\partial \Phi_r}{\partial x}$

$$\partial_x \left((\partial_x \Phi_r)^2 \right) = \frac{\partial V}{\partial x}$$

$$\partial_x \Phi_r = \pm \sqrt{V}$$

- $x = \displaystyle\int \frac{1}{\sqrt{\lambda}\,(\Phi_r^2 - v^2)}\, d\Phi_r = \frac{1}{\sqrt{\lambda}\, v} \operatorname{arctanh}\left(\frac{\Phi_r}{v}\right)$

$$\rightarrow \phi_{DW} = v \cdot \tanh(\sqrt{\lambda}\, v\, x)$$

(2)

- field equation:

$$\partial_\mu \partial^\mu \phi + 2\lambda\,(|\phi|^2 - v^2)\,\phi = 0$$

$\phi \longmapsto \phi + \delta\phi$:

$$\partial_\mu \partial^\mu \phi + \partial_\mu \partial^\mu \delta\phi$$
$$+ 2\lambda\,(|\phi|^2 - v^2 + \phi^* \delta\phi + \phi\,\delta\phi^*)\,(\phi + \delta\phi) = 0$$

Inserting field equation for ϕ gives

$$\partial_\mu \partial^\mu \delta\phi + 2\lambda\left[|\phi|^2 \delta\phi + \phi^2 \delta\phi^* + (|\phi|^2 - v^2)\,\delta\phi\right] = 0$$

- Inserting the DW solution gives

$$\partial_\mu \partial^\mu \delta\phi + 2\lambda v^2\left(\tanh^2(\sqrt{\lambda}\, v x)\,(\delta\phi + \delta\phi^*) + (\tanh^2(\sqrt{\lambda}\, v x) - 1)\;\delta\phi\right) = 0$$

(3)

- $\delta\phi = i\,\gamma(x)\, e^{-i\omega t}$

$$\rightarrow -i\gamma\omega^2 - i\gamma'' + 2\lambda v^2\,(\tanh^2(\sqrt{\lambda}\, v x) - 1)\, i\gamma = 0$$

$$-\gamma'' + 2\lambda v^2\,(\tanh^2(\sqrt{\lambda}\, v x) - 1)\,\gamma = \omega^2 \gamma$$

- The solution to this equation is well-known, but here I will show you how to guess the solution. First let us take the substitution:

$$\gamma \sim \frac{1}{f}$$
$$\gamma' \sim -\frac{f'}{f^2}$$
$$\gamma'' \sim +2\,\frac{f'^2}{f^3} - \frac{f''}{f^2}$$

$$\delta'' \sim +2\frac{f'^2}{f^3} - \frac{f''}{f^2}$$

$$-2\frac{f'^2}{f^3} + \frac{f''}{f^2} + 2\lambda v^2\left(\tanh^2(\sqrt{\lambda}\, v x) - 1\right)\frac{1}{f} = \omega^2 \frac{1}{f}$$

$$-2\frac{f'^2}{f^2} + \frac{f''}{f} + 2\lambda v^2\left(\tanh^2(\sqrt{\lambda}\, v x) - 1\right) = \omega^2$$

- From here the solution can be found very easily. The first term will cancel the tanh-term if

  $$f(x) = \cosh(\sqrt{\lambda}\, v x)$$

  Inserting this gives

  $$-\lambda v^2 = \omega^2$$

  Therefore $\omega^2 < 0$

- There is a mode $\delta\phi = i \frac{1}{\cosh(\sqrt{\lambda}\, v x)} e^{\sqrt{\lambda} v^2 t}$

  that grows in time.

  $\rightarrow$ The domain wall is not stable

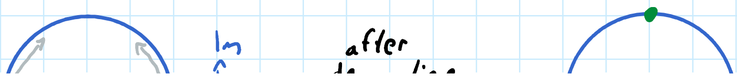
(4)

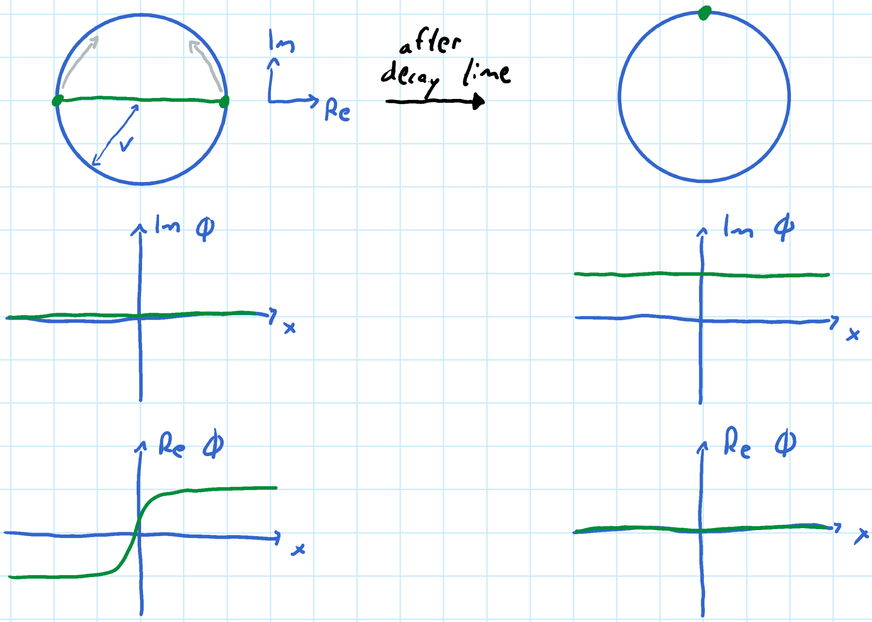
- The vacuum manifold of the theory is a circle S_1.

  $\rightarrow \pi_0(S_1) = 1$

  zeroth homotopy group is trivial and thus stable domain walls are not allowed.

- In the time evolution:





Note that the Goldstone boson is the transmitter of the information about the decay.

(5)

- Vacuum manifold is a circle plus one point

$$\mathcal{M} = S_1 \cup 1$$

$\rightarrow$ $\pi_0 (S_1 \cup 1) = \mathbb{Z}_2$

$\rightarrow$ domain walls are allowed