Problem 1:

(check out Prasad, Sommerfeld - Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon)

$$W_i^a = \varepsilon_{aij} \frac{r^j}{r^2} \frac{1}{g} (1-K(r))$$
$$W_t^a = 0$$
$$\Phi^a = \frac{r^a}{r^2} \frac{1}{g} H(r)$$

(1)

- $$E = \int d^3x \left(\frac{1}{4} G^a_{ij} G^a_{ij} + \frac{1}{2} (D_i\Phi)^a (D_i\Phi)^a + \frac{\lambda}{4} (\Phi^a\Phi^a - v^2)^2 \right)$$

- $$\begin{aligned}(D_i\Phi)^a &= \partial_i \Phi^a + g\, \varepsilon_{abc} W_i^b \Phi^c \\ &= \frac{1}{g}\left(\frac{\delta_{ai}}{r^2} H(r) - 2 \frac{r^a r^i}{r^4} H(r) + \frac{r^a r^i}{r^3} H'(r) \right) \\ &\quad + \frac{1}{g} \underbrace{\varepsilon_{abc}\, \varepsilon_{bij}}_{= \delta_{ci}\delta_{aj} - \delta_{cj}\delta_{ai}} \frac{r^j}{r^2} (1-K(r)) \frac{r^c}{r^2} H(r) \\ &= \frac{1}{g}\left(\cancel{\frac{\delta_{ai}}{r^2} H(r)} - \cancel{2} \frac{r^a r^i}{r^4} H(r) + \frac{r^a r^i}{r^3} H'(r) \right) \\ &\quad + \frac{1}{g} \frac{r^a r^i}{r^4} H(r) (\cancel{1} - K(r)) - \frac{1}{g} \frac{\delta_{ai}}{r^2} H(r) (\cancel{1} - K(r)) \\ &= \frac{1}{g} \frac{1}{r^2} H(r) K(r) \left(\delta_{ai} - \frac{r^a r^i}{r^2} \right) + \frac{1}{g r^2} \frac{r^a r^i}{r^2} (r H'(r) - H(r))\end{aligned}$$

Comment: $\left(\delta_{ai} - \frac{r^a r^i}{r^2}\right)$ is a projector that filters everything out that is proportional to r^a

that filters everything but that is proportional to r^a.

$$\rightarrow (D_i\phi)^a (D_i\phi)^a = \frac{2}{g^2 r^4} H(r)^2 K(r)^2 + \frac{1}{g^2 r^4}\left(r H'(r) - H(r)\right)^2$$

- $G^a_{ij} = \partial_i W^a_j - \partial_j W^a_i + g\,\varepsilon_{abc} W^b_i W^c_j$

$$= \frac{1}{g}\varepsilon_{ajk}\left(\frac{\delta_{ik}}{r^2} - 2\frac{r^k r^i}{r^4}\right)(1-K(r)) - \frac{1}{g}\varepsilon_{ajk}\frac{r^k r^i}{r^3}K'(r)$$

$$- \frac{1}{g}\varepsilon_{aik}\left(\frac{\delta_{jk}}{r^2} - 2\frac{r^k r^j}{r^4}\right)(1-K(r)) + \frac{1}{g}\varepsilon_{aik}\frac{r^k r^j}{r^3}K'(r)$$

$$+ \frac{1}{g}\underbrace{\varepsilon_{abc}\,\varepsilon_{bik}}_{=\delta_{ci}\delta_{ka} - \delta_{ck}\delta_{ia}}\frac{r^k}{r^2}\varepsilon_{cjm}\frac{r^m}{r^2}(1-K(r))^2$$

$$= \frac{1}{g}\varepsilon_{aji}\frac{1}{r^2} - \frac{2}{g}\frac{1}{r^2}\left(\varepsilon_{ajk}\frac{r^k r^i}{r^2} - \varepsilon_{aik}\frac{r^k r^j}{r^2}\right)(1-K(r))$$

$$- \frac{1}{g}\frac{1}{r}\left(\varepsilon_{ajk}\frac{r^k r^i}{r^2} - \varepsilon_{aik}\frac{r^k r^j}{r^2}\right)K'(r)$$

$$+ \frac{1}{g}\frac{1}{r^2}\varepsilon_{ijk}\frac{r^k r^a}{r^2}(1-K(r))^2$$

- For the green terms we can use a trick

$$\varepsilon_{abk}\hat{r}^k\hat{r}^c + \varepsilon_{bck}\hat{r}^k\hat{r}^a + \varepsilon_{cak}\hat{r}^k\hat{r}^b = \varepsilon_{abc}$$

$$G^a_{ij} = -\frac{1}{g}\cancel{\varepsilon_{aij}}\frac{1}{r^2} - \frac{2}{g}\frac{1}{r^2}\left(-\varepsilon_{jik}\frac{r^k r^a}{r^2} + \cancel{\varepsilon_{aji}}\right)(1-\cancel{K}(r))$$

$$- \frac{1}{g}\frac{1}{r}\left(-\varepsilon_{jik}\frac{r^k r^a}{r^2} + \varepsilon_{aji}\right)K'(r)$$

$$+ \frac{1}{g}\frac{1}{r^2}\varepsilon_{ijk}\frac{r^k r^a}{r^2}\left(\cancel{1} - 2\cancel{K}(r) + K(r)^2\right)$$

$$= \frac{1}{gr^2}\, \varepsilon_{ijk} \frac{r^k r^a}{r^2} \left(K(r)^2 - 1\right) + \frac{1}{gr}\, \varepsilon_{ijk} \underbrace{\left(\delta_{ak} - \frac{r^k r^a}{r^2}\right)}_{\text{remember this is a projector}} K'(r)$$

- $$G^a_{ij} G^a_{ij} = \frac{2}{g^2 r^4} \left(K(r)^2 - 1\right)^2 + \frac{1}{g^2 r^2}\, 2 \underbrace{\left(\delta_{kk} - \frac{r^k r^k}{r^2}\right)}_{=2} K'(r)^2$$

- $$\mathcal{E} = + \frac{1}{4} G^a_{ij} G^a_{ij} + \frac{1}{2} (D_i \Phi)^a (D_i \Phi)^a + \frac{\lambda}{4} \left(\Phi^a \Phi^a - v^2\right)^2$$

$$= \frac{1}{g^2 r^2} K'(r)^2 + \frac{1}{2 g^2 r^4} \left(K(r)^2 - 1\right)^2 + \frac{1}{g^2 r^4} H(r)^2 K(r)^2 + \frac{1}{2} \frac{1}{g^2 r^4} \left(r H' - H\right)^2$$

$$+ \frac{\lambda}{4} \left(\frac{1}{g^2 r^2} H(r)^2 - v^2\right)^2$$

$$= \frac{1}{g^2 r^2} \left[K'(r)^2 + \frac{\left(1 - K(r)^2\right)^2}{2 r^2} + \frac{H(r)^2 K(r)^2}{r^2} + \frac{\left(r H'(r) - H(r)\right)^2}{2 r^2} + \frac{\lambda v^4}{4} \left(\frac{H(r)^2}{m_v^2 r^2} - 1 \right)^2 \right]$$

$m_v = vg$

$$H' - \frac{H}{r} = r \left(\frac{H}{r}\right)'$$

- The energy is finite if the expression inside the bracket vanishes for $r \to \infty$. $\left(E = 4\pi \int dr\, r^2 \mathcal{E}\right)$

We obtain the limits

$$\frac{H(r)}{m_v r} \xrightarrow{r \to \infty} 1 \quad , \quad K(r) \xrightarrow{r \to \infty} 0$$

Furthermore, we need

$$K(r) \xrightarrow{r \to 0} 1 \quad , \quad \frac{H(r)}{m_v r} \xrightarrow{r \to 0} 0$$

$$K'(r) \xrightarrow{r \to 0} 0$$

(2)

- field equations

$$(D_\mu G^{\mu\nu})^a = -g\,\varepsilon_{abc}\,\Phi^b (D^\nu \Phi)^c$$
$$(D_\mu D^\mu \Phi)^a + \lambda\,(\Phi^b\Phi^b - v^2)\,\Phi^a = 0$$

- Let us start with the first equation:

$$0 = (D_i G_{ij})^a - g\,\varepsilon_{abc}\,\Phi^b (D_i\Phi)^c$$
$$= \partial_i G^a_{ij} + g\,\varepsilon_{abc} W_i^b G^c_{ij} - g\,\varepsilon_{abc}\,\Phi^b (D_i\Phi)^c$$
$$= \frac{1}{gr^4}\varepsilon_{ijk} r^k \delta_{ai} (K(r)^2 - 1) - \frac{1}{gr^3}\cancel{\varepsilon_{ija} r^i} K'(r) + \frac{1}{gr^2}\varepsilon_{ija} r^i K''(r)$$
$$- \frac{1}{gr^3}\cancel{\varepsilon_{ijk} r^k}\delta_{ai} K'(r)$$
$$+ \frac{1}{gr^6}\underbrace{\varepsilon_{abc}\,\varepsilon_{bim}}_{=\cancel{\delta_{ci}\delta_{am}} - \delta_{cm}\delta_{ai}}\varepsilon_{ijk}\, r^m r^k r^c (1-K(r))(K(r)^2-1)$$
$$+ \frac{1}{gr^3}\varepsilon_{abc}\,\varepsilon_{bim}\,\varepsilon_{ijk}\,\cancel{r^m\left(\delta_{ck} - \frac{r^k r^c}{r^2}\right)}^{=0} K'(r)(1-K(r))$$
$$- \frac{1}{gr^4}\varepsilon_{abj}\, r^b H(r)^2 K(r)$$

$$\rightarrow K''(r) + \frac{1}{r^2}(1-K(r)^2) - \frac{1}{r^2}(1-K(r))(1-K(r)^2) - \frac{1}{r^2}H(r)^2K(r) = 0$$
$$\rightarrow K''(r) = \frac{1}{r^2}\left(K(r)^3 - K(r) + H(r)^2 K(r)\right)$$

- $(D_\mu D^\mu \Phi)^a + \lambda\,(\Phi^b\Phi^b - v^2)\,\Phi^a = 0$

$$0 = -\partial_i (D_i\Phi)^a - g\,\varepsilon_{abc} W_i^b (D_i\Phi)^c + \lambda\,(\Phi^b\Phi^b - v^2)\,\Phi^a$$
$$= +\frac{1}{gr^2} H(r)K(r)\underbrace{\left(\frac{\delta_{ai} r^i}{r^2} + \frac{r^a\delta_{ii}}{r^2} - 2\frac{r^a r^i r^i}{r^4}\right)}_{=2\frac{r^a}{r^2}}$$
$$- \frac{1}{\;}\left(-4\frac{r^a r^i r^i}{\;} + \frac{\delta_{ai} r^i}{\;} + \frac{r^a\delta_{ii}}{\;}\right)(r H'(r) - H(r))$$

$$- \frac{1}{g} \underbrace{\left(-4 \frac{r^a r^i r^i}{r^6} + \frac{\delta_{ai} r^i}{r^4} + \frac{r^a \delta_{ii}}{r^4} \right)}_{=0} \left(r H'(r) - H(r) \right)$$

$$- \frac{1}{g r^4} r^a r^i \left(\frac{r^i}{r} \cancel{H'(r)} + r^i H''(r) - \frac{r^i}{r} \cancel{H'(r)} \right)$$

$$- \frac{2}{g r^4} r^a (1 - K(r)) H(r) K(r)$$

$$+ \lambda \left(\frac{1}{g^2 r^2} H(r)^2 - v^2 \right) \frac{r^a}{r^2} \frac{1}{g} H(r)$$

$$\rightarrow H''(r) - \frac{2}{r^2} H(r) K(r)^2 - \lambda v^2 H(r) \left(\frac{H(r)^2}{v^2 g^2 r^2} - 1 \right) = 0$$

$$\rightarrow H''(r) = \frac{2}{r^2} H(r) K(r)^2 + \lambda v^2 H(r) \left(\frac{H(r)^2}{m_v^2 r^2} - 1 \right)$$

(3)

- $$H'(r) = \frac{m_v}{\tanh(m_v r)} - \frac{m_v r}{\sinh^2(m_v r)} \cdot m_v$$

$$H''(r) = - \frac{2 m_v^2}{\sinh^2(m_v r)} + \frac{2 \cosh(m_v r)}{\sinh^3(m_v r)} m_v^3 r$$

$$K'(r) = \frac{m_v}{\sinh(m_v r)} - \frac{\cosh(m_v r)}{\sinh^2(m_v r)} m_v^2 r$$

$$K''(r) = -2 \frac{\cosh(m_v r)}{\sinh^2(m_v r)} m_v^2 - \frac{m_v^3 r}{\sinh(m_v r)} + 2 \frac{\cosh^2(m_v r)}{\sinh^3(m_v r)} m_v^3 r$$

- Check first equation:

$$- \frac{2 m_v^2}{\sinh^2(m_v r)} + \frac{2 \cosh(m_v r)}{\sinh^3(m_v r)} m_v^3 r = \frac{2}{r^2} \left(\frac{\cosh(m_v r)}{\sinh(m_v r)} \cdot m_v r - 1 \right) \frac{m_v^2 r^2}{\sinh^2(m_v r)}$$

$$-1 + \frac{\cosh(m_v r)}{\sinh(m_v r)} m_v r = \frac{\cosh(m_v r)}{\sinh(m_v r)} m_v r - 1$$

✓

- Check second equation:

$$\cosh(m_v r) \quad m_v^3 r \quad \cosh^2(m_v r)$$

- Check second equation:

$$-2\,\frac{\cosh(m_v r)}{\sinh^2(m_v r)}\,m_v^2 - \frac{m_v^3 r}{\sinh(m_v r)} + 2\,\frac{\cosh^2(m_v r)}{\sinh^3(m_v r)}\,m_v^3 r$$

$$= \frac{1}{r^2}\left(\frac{(m_v r)^3}{\sinh^3(m_v r)} - \frac{m_v r}{\sinh(m_v r)} + \left(\frac{\cosh(m_v r)}{\sinh(m_v r)}\cdot m_v r - 1\right)^2 \frac{m_v r}{\sinh(m_v r)}\right)$$

$$-2\,\cancel{\frac{\cosh(m_v r)}{\sinh(m_v r)}}\,m_v - m_v^2 r + \cancel{2}\,\frac{\cosh^2(m_v r)}{\sinh^2(m_v r)}\,m_v^2 r$$

$$= \frac{m_v^2 r}{\sinh^2(m_v r)} - \cancel{\frac{m_v}{r}} + \cancel{\frac{\cosh^2(m_v r)}{\sinh^2(m_v r)}}\,m_v^2 r - 2\,\cancel{\frac{\cosh(m_v r)}{\sinh(m_v r)}}\,m_v + \cancel{\frac{m_v}{r}}$$

$$-1 + \frac{\cosh^2(m_v r)}{\sinh^2(m_v r)} = \frac{1}{\sinh^2(m_v r)} \qquad \checkmark$$

(4)

- $E = \int d^3x \left[\frac{1}{4} G_{ij}^a G_{ij}^a + \frac{1}{2}(D_i\Phi)^a (D_i\Phi)^a + V(\Phi)\right]$ with $B_i^a = \frac{1}{2}\varepsilon_{ijk} G_{jk}^a$

$$= \int d^3x \left[\frac{1}{2} B_i^a B_i^a + \frac{1}{2}(D_i\Phi)^a(D_i\Phi)^a - B_i^a (D_i\Phi)^a + B_i^a (D_i\Phi)^a + V(\Phi)\right]$$

$$= \int d^3x\, B_i^a (D_i\Phi)^a + \int d^3x \left[\frac{1}{2}\left(B_i^a - (D_i\Phi)^a\right)^2 + V(\Phi)\right]$$

- Bianchi identity: $\varepsilon_{ijk}(D_i G_{jk})^a = 0$ with $B_i^a = \frac{1}{2}\varepsilon_{ijk} G_{jk}^a$

$$D_i B_i^a = 0$$

$$\int d^3x\, B_i^a (D_i\Phi)^a = \int d^3x \left(B_i^a\, \partial_i \Phi^a + g\, B_i^a\, \varepsilon_{abc} W_i^b \Phi^c\right)$$

int by parts

$$= \int d^3x \left(\partial_i (B_i^a \Phi^a) - \Phi^a \underbrace{D_i B_i^a}_{=0}\right)$$

$$= \int_{\partial V} dS_i \cdot \underbrace{B_i^a \hat{\Phi}^a}_{=\frac{1}{g}\frac{r^i}{r^3}} \cdot |\Phi| \quad \text{at } \partial V$$

$$= \frac{\hat{r}}{g}\frac{1}{r^3} \quad \text{at } \partial V$$

$$= \int d\Phi \, d\Theta \, r^2 \frac{r^i}{r} \frac{1}{g} \frac{r^i}{r^3} \cdot v$$

$$= \frac{4\pi v}{g}$$

$$\rightarrow E = \frac{4\pi v}{g} + \int d^3x \left[\frac{1}{2} (B_i^a - (D_i\Phi)^a)^2 + V(\Phi) \right]$$

(5)

- In the BPS limit $V(\Phi) \to 0$.

Since the monopole is a static solution of the field equations, it has to minimize the energy. The energy is minimized for

$$B_i^a = (D_i \Phi)^a$$

The mass of the monopole in that case is

$$M_{BPS} = \frac{4\pi v}{g}$$

Problem 2:

(1)

- $\hat{\Phi} \longmapsto U\hat{\Phi}U^\dagger \quad , \quad W_\mu \longmapsto UW_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger$

$$A_\mu = \hat{\Phi}^a W_\mu^a = 2\,\mathrm{Tr}\,(\hat{\Phi} W_\mu)$$

$$\longmapsto 2\,\mathrm{Tr}\left(U\hat{\Phi}U^\dagger U W_\mu U^\dagger + \frac{i}{g} U\hat{\Phi}U^\dagger U(\partial_\mu U^\dagger)\right)$$

$$= 2\,\mathrm{Tr}\,(\hat{\Phi} W_\mu) + 2\frac{i}{g}\,\mathrm{Tr}\left(\hat{\Phi}(\partial_\mu U^\dagger)U\right)$$

$$= A_\mu + \frac{2i}{g}\,\mathrm{Tr}\,(\hat{\Phi}(\partial_\mu U^\dagger)U)$$

- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\longmapsto F_{\mu\nu} + \frac{2i}{g} Tr((\partial_\mu \hat{\Phi})(\partial_\nu U^\dagger) U) + \frac{2i}{g} Tr(\cancel{\hat{\Phi}(\partial_\mu \partial_\nu U^\dagger) U})$$
$$Tr(\hat{\Phi}(\partial_\nu U^\dagger)(\partial_\mu U)) - (\mu \leftrightarrow \nu)$$

$$\neq F_{\mu\nu}$$

(2)

- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{g} \varepsilon_{abc} \hat{\Phi}^a (\partial_\mu \hat{\Phi}^b)(\partial_\nu \hat{\Phi}^c)$

$$= (\partial_\mu \hat{\Phi}^a) W_\nu^a + \underline{\hat{\Phi}^a \partial_\mu W_\nu^a} - (\partial_\nu \hat{\Phi}^a) W_\mu^a - \underline{\hat{\Phi}^a \partial_\nu W_\mu^a}$$
$$\underline{- \frac{1}{g} \varepsilon_{abc} \hat{\Phi}^a (\partial_\mu \hat{\Phi}^b)(\partial_\nu \hat{\Phi}^c)} + \underline{g \hat{\Phi}^a \varepsilon_{abc} W_\mu^b W_\nu^c}$$
$$\underline{- g \hat{\Phi}^a \varepsilon_{abc} W_\mu^b W_\nu^c}$$

$$= \hat{\Phi}^a G_{\mu\nu}^a + (\partial_\mu \hat{\Phi}^a) W_\nu^a - (\partial_\nu \hat{\Phi}^a) W_\mu^a$$
$$- \frac{1}{g} \varepsilon_{abc} \hat{\Phi}^a (D_\mu \hat{\Phi})^b (D_\nu \hat{\Phi})^c$$
$$+ \frac{1}{g} \varepsilon_{abc} \hat{\Phi}^a (\partial_\mu \hat{\Phi}^b) \; g \varepsilon_{cde} W_\nu^d \hat{\Phi}^e$$
$$+ \frac{1}{g} \varepsilon_{abc} \hat{\Phi}^a \, g \varepsilon_{bde} W_\mu^d \hat{\Phi}^e \; (\partial_\nu \hat{\Phi}^c)$$

$$= \hat{\Phi}^a G_{\mu\nu}^a - \frac{1}{g} \varepsilon_{abc} \hat{\Phi}^a (D_\mu \hat{\Phi})^b (D_\nu \hat{\Phi})^c$$
$$+ \cancel{(\partial_\mu \hat{\Phi}^a)} W_\nu^a - \cancel{(\partial_\nu \hat{\Phi}^a)} W_\mu^a$$
$$- \hat{\Phi}^a \cancel{(\partial_\mu \hat{\Phi}^b)} W_\nu^b \hat{\Phi}^a + \hat{\Phi}^a W_\mu^b \cancel{\hat{\Phi}^a} (\partial_\nu \hat{\Phi}^b)$$

$$= \hat{\Phi}^a G_{\mu\nu}^a - \frac{1}{g} \varepsilon_{abc} \hat{\Phi}^a (D_\mu \hat{\Phi})^b (D_\nu \hat{\Phi})^c$$

(3)

- $F_{\mu\nu} = 2\, Tr(\hat{\Phi} G_{\mu\nu}) - \frac{2}{g} Tr(\hat{\Phi} [D_\mu \hat{\Phi}, D_\nu \hat{\Phi}])$

$$\hat{\Phi} \longmapsto U \hat{\Phi} U^\dagger$$

$$G_{\mu\nu} \longmapsto U G_{\mu\nu} U^\dagger$$

$$(D_\mu \hat{\Phi}) \longmapsto U (D_\mu \hat{\Phi}) U^\dagger$$

$G_{\mu\nu} \longmapsto U G_{\mu\nu} U^{+}$

$(D_\mu \hat{\Phi}) \longmapsto U (D_\mu \hat{\Phi}) U^{+}$

$\rightarrow$ $F_{\mu\nu}$ is now gauge invariant

(4)

- $\hat{\Phi}^a = \delta_{a3}$

$$\rightarrow F_{\mu\nu} = G^3_{\mu\nu} - \frac{1}{g} \varepsilon_{3bc} (g \varepsilon_{bd3} W^d_\mu) \cdot (g \varepsilon_{ce3} W^e_\nu)$$
$$= \partial_\mu W^3_\nu - \partial_\nu W^3_\mu + g \varepsilon_{3bc} W^b_\mu W^c_\nu$$
$$- g (\delta_{cd} - \cancel{\delta_{c3}} \delta_{d3}) \varepsilon_{ce3} W^d_\mu W^e_\nu$$

$$= \partial_\mu W^3_\nu - \partial_\nu W^3_\mu + \cancel{g \varepsilon_{3bc} W^b_\mu W^c_\nu}$$
$$\cancel{- g \varepsilon_{3ce} W^c_\mu W^e_\nu}$$

$$= \partial_\mu W^3_\nu - \partial_\nu W^3_\mu$$

Problem 3:

(1)

- $\Phi^a(x) \longmapsto \Phi^a(x - a(t)) \overset{\text{Taylor}}{\approx} \Phi^a(x) - a_i(t)\, \partial_i\, \Phi^a(x)$

$W^a_i(x) \longmapsto W^a_i(x - a(t)) \approx W^a_i(x) - a_j(t)\, \partial_j\, W^a_i(x)$

- To satisfy the Gauss constraint we need to transform also $W^a_t \longmapsto W'^a_t$. It is convenient to keep $W^a_t = 0$ and therefore, we will apply a further gauge transformation without violating the Gauss constraint.

- As already shown on problem sheet 7, the

- As already shown on problem sheet 7, the infinitesimal gauge transformation is given by

$$\Phi^a(x) \longmapsto \Phi^a(x) + \varepsilon_{abc}\, \alpha_b(x)\, \Phi^c(x)$$

$$W_i^a(x) \longmapsto W_i^a(x) + \varepsilon_{abc}\, \alpha_b(x)\, W_i^c(x) - \frac{1}{g}\partial_i \alpha_a(x)$$

- Combining this with the space translation gives the transformation on the sheet.

(2)

- for $\alpha_a = g\, a_i(t)\, W_i^a$ we obtain

$$\delta\Phi^a = g\,\varepsilon_{abc}\, a_i(t)\, W_i^b\, \Phi^c + a_i(t)\, \partial_i \Phi^a$$
$$= a_i(t)\, (D_i\Phi)^a$$

$$\delta W_i^a = g\,\varepsilon_{abc}\, a_j(t)\, W_j^b\, W_i^c - a_j(t)\, \partial_i W_j^a + a_j\, \partial_j W_i^a$$
$$= a_j(t)\, G_{ji}^a$$

- the time derivative is

$$\partial_t f = \lim_{\Delta t \to 0} \frac{f(x - a(\Delta t)) - f(x)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\delta f(\Delta t)}{\Delta t} = \partial_t\, \delta f(\Delta t)$$

- the kinetic energy becomes

$$T = \int d^3x\ \frac{1}{2}\left(\partial_t W_i^a\, \partial_t W_i^a + \partial_t \Phi^a\, \partial_t \Phi^a\right)$$
$$= \int d^3x\ \frac{1}{2}\left(\dot a_j\, G_{ji}^a\, \dot a_k\, G_{ki}^a + \dot a_i\, (D_i\Phi)^a\, \dot a_j\, (D_j\Phi)^a\right)$$

$$= \frac{1}{2}\, \dot a_i\, \dot a_j \int d^3x \left(G_{ik}^a\, G_{jk}^a + (D_i\Phi)^a\, (D_j\Phi)^a\right)$$

(3)

- $\underbrace{\int d^3x\ G_{ik}^a\, G_{jk}^a}_{\text{sym. in } i,j} = \delta_{ij} \cdot N$

$\underbrace{\quad}_{\text{sym. in } i,j}$

What is $\mathcal{N}$? Let us take $i,j = x$:

$$\mathcal{N} = \int d^3x \; G^a_{xk} G^a_{xk}$$

- $G^a_{ik} G^a_{jk}$

$$= \varepsilon_{ikm} B^a_m \; \varepsilon_{jkn} B^a_n$$

$$= \delta_{ij} B^a_k B^a_k - B^a_i B^a_j$$

$$G^a_{xk} G^a_{xk} = B^a_y B^a_y + B^a_z B^a_z = 2 B^a_x B^a_x = G^a_{yk} G^a_{yk} = G^a_{zk} G^a_{zk}$$

we know that the magnetic field spherically symmetric

$$\rightarrow \quad G^a_{xk} G^a_{xk} = \frac{1}{3} G^a_{mk} G^a_{mk}$$

$$\rightarrow \quad \int d^3x \; G^a_{ik} G^a_{jk} = \frac{1}{3} \delta_{ij} \int d^3x \; G^a_{mk} G^a_{mk}$$

- $\int d^3x \; (D_i \Phi)^a (D_j \Phi)^a = \mathcal{N} \delta_{ij}$

What is $\mathcal{N}$?

In BPS limit $(D_i \Phi)^a = B^a_i$

For $i,j = x$ we have:

$$\mathcal{N} = \int d^3x \; (D_x \Phi)^a (D_x \Phi)^a = \int d^3x \; B^a_x B^a_x = \frac{1}{3} \int d^3x \; B^a_k B^a_k$$

- In the BPS limit we have $G^a_{mk} G^a_{mk} = 2 (D_k \Phi)^a (D_k \Phi)^a$ (1)

- $T = \frac{1}{2} \dot{a}_i \dot{a}_j \int d^3x \left(\underbrace{\frac{1}{3} G^a_{mk} G^a_{mk}} + \frac{1}{3} (D_k \Phi)^a (D_k \Phi)^a \right)$

$$= \frac{1}{4} G^a_{mk} G^a_{mk} + \underbrace{\frac{1}{12} G^a_{mk} G^a_{mk}}$$

$$= \frac{1}{4} G^a_{mk} G^a_{mk} + \underbrace{\frac{1}{12} G^a_{mk} G^a_{mk}}_{\overset{(1)}{=} \frac{1}{6} (D_k \Phi)^a (D_k \Phi)^a}$$

$$= \frac{1}{2} \dot{a}_i \dot{a}_i \int d^3x \left(\frac{1}{4} G^a_{mk} G^a_{mk} + \frac{1}{2} (D_k \Phi)^a (D_k \Phi)^a \right)$$

$$= \frac{1}{2} M \dot{a}_i \dot{a}_i$$

(4)

- $\Phi \longmapsto U \Phi U^{+} = U U^{+} \Phi = \Phi \quad \rightarrow \delta \Phi^a = 0$

- $$\delta W^a_\mu = \varepsilon_{abc} \Phi^b \frac{\chi}{v} W^c_\mu - \frac{1}{g} \partial_\mu \left(\Phi^a \frac{\chi}{v} \right)$$
$$= \varepsilon_{abc} \Phi^b W^c_\mu \frac{\chi}{v} - \frac{1}{g} (\partial_\mu \Phi^a) \frac{\chi}{v} - \frac{1}{g} \Phi^a \partial_\mu \frac{\chi}{v}$$
$$= - \frac{1}{gv} \chi (D_\mu \Phi)^a - \frac{1}{gv} \Phi^a \partial_\mu \chi$$

(5)

- The kinetic energy is
$$T = \int d^3x \left(\frac{1}{2} (\partial_t W^a_i)^2 + \frac{1}{2} (\partial_t \Phi^a)^2 \right)$$
$$= \frac{1}{2} \int d^3x \frac{1}{g^2 v^2} (D_i \Phi)^a (D_i \Phi)^a \dot{\chi}^2$$
$$\overset{(1)}{=} \frac{1}{2} \int d^3x \frac{1}{g^2 v^2} \left(\frac{1}{4} G^a_{ij} G^a_{ij} + \frac{1}{2} (D_i \Phi)^a (D_i \Phi)^a \right) \dot{\chi}^2$$
$$= \frac{1}{2} \frac{1}{g^2 v^2} M \dot{\chi}^2$$

(6)

- $$T = \frac{1}{2} M \delta_{ij} \dot{a}_i \dot{a}_j + \frac{1}{2} M \frac{1}{g^2 v^2} \dot{\chi}^2$$

In coordinates (a_x, a_y, a_z, χ), the metric is

$$g = \mathrm{diag}\left(1, 1, 1, \frac{1}{\ldots}\right)$$

$$g_{ij} = \mathrm{diag}\left(1, 1, 1, \frac{1}{g^2 v^2}\right)$$

→ the moduli space is flat

(7)

- The $U(1)$ electric field is given by

$$E_i = \hat{\Phi}^a E_i^a$$
$$= \hat{\Phi}^a G_{0i}^a$$
$$= \hat{\Phi}^a \left(\partial_t W_i^a - \partial_i \underbrace{W_t^a}_{=0} + g\,\varepsilon_{abc} \underbrace{W_t^b}_{=0} W_i^c\right)$$
$$= \hat{\Phi}^a \partial_t W_i^a$$
$$= -\hat{\Phi}^a \frac{1}{gv} \dot{\chi} (D_i \Phi)^a \qquad \text{with} \quad B_i^a = (D_i \Phi)^a$$
$$= -\frac{1}{gv} \dot{\chi} B_i$$

- $$Q_e = \int d^3x\, \partial_i E_i$$
$$= -\frac{\lambda}{gv} \int d^3x\, \partial_i B_i$$
$$= -\frac{\lambda}{gv} \underbrace{Q_m}_{=\frac{4\pi}{g}}$$