Problem 1:

$$\mathcal{L} = \mathrm{Tr}\left(\left(\partial_\mu \Phi^\dagger\right)\left(\partial^\mu \Phi\right)\right) - \lambda\left(\mathrm{Tr}(\Phi\Phi) - \frac{v^2}{2}\right)^2$$

(1)

- $\mathrm{Tr}(\Phi^\dagger\Phi) = \Phi^a\Phi^b\,\mathrm{Tr}(T^aT^b) = \frac{1}{2}\Phi^a\Phi^a$

$$\rightarrow V = \frac{\lambda}{4}\left(\Phi^a\Phi^a - v^2\right)^2$$

Minima are at $\Phi^a\Phi^a = v^2$.

Since Φ^a has three components, the equation $\Phi^a\Phi^a = v^2$ describes a 2-sphere.

(2)

- vev: $\langle\Phi\rangle = v\cdot T^3$

- How the generators act on Φ?

$$\begin{aligned}\Phi \longmapsto\ & U\Phi U^\dagger \\ =\ & e^{i\varphi_a T_a}\,\Phi^b T^b\, e^{-i\varphi_c T_c} \\ =\ & \Phi^b T^b + i\varphi_a \Phi^b T_a T_b - i\varphi_c \Phi^b T_b T_c + \mathcal{O}(\varphi^2) \\ =\ & \Phi + i\varphi_a\left[T_a, \Phi\right]\end{aligned}$$

- $\langle\Phi\rangle \longmapsto \langle\Phi\rangle + i\varphi_a\left[T_a, \langle\Phi\rangle\right]$

The vacuum expectation value is invariant if

$$\left[T_a, \langle\Phi\rangle\right] = 0$$

For $\langle\Phi\rangle = vT_3$, the unbroken generator is T_3, For $\langle\Phi\rangle = v\,T_3$, the unbroken generator is T_3, which means that transformations by $U = e^{i\varphi T_3}$ are a symmetry, but transformations by $U = e^{i\varphi T_{1,2}}$ are not.

$\rightarrow$ The remaining symmetry is $U(1)$.

(3)

- $\Phi \longrightarrow (v+h)T^3 + \varphi_1 T^1 + \varphi_2 T^2 \quad \rightarrow \quad \Phi^a = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ v+h \end{pmatrix}$

$$V = \frac{\lambda}{4}\left(\Phi^a\Phi^a - v^2\right)^2$$

$$= \frac{\lambda}{4}\left(\varphi_1^2 + \varphi_2^2 + \cancel{v^2} + 2vh + h^2 - \cancel{v^2}\right)^2 \qquad (1)$$

keep only mass terms

$$\supset \lambda v^2 h^2$$

- We obtained two massless fields φ_1, φ_2 and one massive field h with mass $m_h = \sqrt{2\lambda}\,v$

- From (1) we can see the remaining $U(1)$ symmetry: $h \longmapsto h$

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \longmapsto O \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \quad \text{with} \quad O \in SO(2) \simeq U(1)$$

(4)

- $\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\left(G_{\mu\nu}G^{\mu\nu}\right) + \mathrm{Tr}\left((D_\mu\Phi)^\dagger(D^\mu\Phi)\right) - \lambda\left(\mathrm{Tr}(\Phi^\dagger\Phi) - \frac{v^2}{2}\right)^2$

- $D_\mu\langle\Phi\rangle = \partial_\mu\langle\Phi\rangle - ig\left[W_\mu, vT^3\right]$

$$= -ivg\,\varepsilon_{ab3}\,W_\mu^b\,T^a$$

$$= -ivg\, \varepsilon_{ab3}\, W_\mu^b T^a$$
$$= -ivg\, (W_\mu^2 T^1 - W_\mu^1 T^2)$$

- The mass terms for the gauge fields are coming from

$$\mathrm{Tr}\left[(D_\mu\langle\Phi\rangle)^\dagger (D^\mu\langle\Phi\rangle)\right]$$
$$= v^2 g^2\, \mathrm{Tr}\big(W_\mu^2 W_2^\mu T^1 T^1 + W_\mu^1 W_1^\mu T^2 T^2$$

$\mathrm{Tr}(T^a T^b) = \frac{\delta_{ab}}{2}$

$$+ W_\mu^1 W_2^\mu T^1 T^2 + W_\mu^2 W_1^\mu T^2 T^1 \big)$$
$$= \frac{1}{2} v^2 g^2 \, (W_\mu^1 W_1^\mu + W_\mu^2 W_2^\mu)$$

→ We obtained two massive fields W_μ^1, W_μ^2 with mass $m_V = v \cdot g$ and one massless field W_μ^3

Problem 2:

(1)

- $$\mathcal{J}^\mu = -\frac{1}{8\pi}\, \varepsilon^{\mu\alpha\beta\gamma}\, \varepsilon_{abc}\, \partial_\alpha \hat{\Phi}^a\, \partial_\beta \hat{\Phi}^b\, \partial_\gamma \hat{\Phi}^c$$

$$\partial_\mu \mathcal{J}^\mu = -\frac{1}{8\pi}\, \varepsilon^{\mu\alpha\beta\gamma}\, \varepsilon_{abc} \big[(\partial_\mu \partial_\alpha \hat{\Phi}^a)\, \partial_\beta \hat{\Phi}^b\, \partial_\gamma \hat{\Phi}^c$$
$$+ \partial_\alpha \hat{\Phi}^a\, (\partial_\mu \partial_\beta \hat{\Phi}^b)\, \partial_\gamma \hat{\Phi}^c$$
$$+ \partial_\alpha \hat{\Phi}^a\, \partial_\beta \hat{\Phi}^b\, (\partial_\mu \partial_\gamma \hat{\Phi}^c) \big] = 0,$$

because $$\underbrace{\varepsilon^{\mu\alpha\beta\gamma}}_{\substack{\text{anti-sym.}\\ \text{in } \mu,\alpha}}\ \underbrace{\partial_\mu \partial_\alpha}_{\substack{\text{symmetric}\\ \text{in } \mu,\alpha}} = 0$$

(2)

- $$\hat{w} = \begin{pmatrix} \cos(n\varphi)\ \sin\Theta \\ \sin(n\varphi)\ \sin\Theta \\ \cos\Theta \end{pmatrix}$$

$$\varphi = \arctan\left(\frac{y}{x}\right), \quad \Theta = \arccos\left(\frac{z}{r}\right)$$

- See the solution in the Mathematica notebook.

(3)

- $$\begin{aligned} Q &= \int d^3x\ J^0 \\ &= -\frac{1}{8\pi}\int d^3x\ \varepsilon^{0ijk}\,\varepsilon_{abc}\,(\partial_i\hat{\Phi}^a)(\partial_j\hat{\Phi}^b)(\partial_k\hat{\Phi}^c) \\ &= +\frac{1}{8\pi}\int d^3x\ \varepsilon_{ijk}\,\varepsilon_{abc}\,\partial_i\left(\hat{\Phi}^a(\partial_j\hat{\Phi}^b)(\partial_k\hat{\Phi}^c)\right) \\ &\overset{(2)}{=} +\frac{1}{8\pi}\int d^3x\ \varepsilon_{ijk}\,\partial_i\left(n\,\varepsilon_{jkm}\frac{r^m}{r^3}\right) \\ &= +\frac{n}{4\pi}\int d^3x\ \partial_i\left(\frac{r^i}{r^3}\right) \\ &= +\frac{1}{4\pi}\underbrace{\int dS^i\ \frac{r^i}{r^3}}_{=4\pi} = n \end{aligned}$$

Problem 3:

(1)

- $$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\operatorname{Tr}(G_{\mu\nu}G^{\mu\nu}) + \operatorname{Tr}\left((D_\mu\Phi)^\dagger(D^\mu\Phi)\right) - \lambda\left(\operatorname{Tr}(\Phi^\dagger\Phi) - \frac{v^2}{2}\right)^2 \\ &= -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + \frac{1}{2}(D_\mu\Phi)^a(D^\mu\Phi)^a - \frac{\lambda}{4}(\Phi^a\Phi^a - v^2)^2 \\ &= +\frac{1}{2}\underbrace{G^a_{0i}}_{=0}G^a_{0i} - \frac{1}{4}G^a_{ij}G^a_{ij} + \frac{1}{2}\underbrace{(D_0\Phi)^a}_{=0}(D_0\Phi)^a - \frac{1}{2}(D_i\Phi)^a(D_i\Phi)^a \\ &\quad - \frac{\lambda}{4}(\Phi^a\Phi^a - v^2)^2 \end{aligned}$$

$$\to \mathcal{E} = \frac{1}{4} G^a_{ij} G^a_{ij} + \frac{1}{2} (D_i\Phi)^a (D_i\Phi)^a + \frac{\lambda}{4} (\Phi^a\Phi^a - v^2)^2$$

(2)

- $$(D_i\vec{\Phi}) = \partial_i \vec{\Phi} + g \vec{W}_i \times \vec{\Phi}$$
$$\xrightarrow{r\to\infty} v\, \partial_i \hat{w} + vg\, \vec{W}_i \times \hat{w} \stackrel{!}{=} 0$$

- Ansatz: $$\vec{W}_i \xrightarrow{r\to\infty} \frac{1}{g} (\partial_i \hat{w}) \times \hat{w} + c_i \hat{w}$$

$$v\, \partial_i \hat{w} + v\, (\partial_i \hat{w} \times \hat{w}) \times \hat{w} + vg\, c_i \underbrace{\hat{w} \times \hat{w}}_{=0} = 0$$

$$v\, \partial_i \hat{w} - v\, \partial_i \hat{w} \underbrace{(\hat{w}\cdot\hat{w})}_{=1} + v\, \underbrace{(\hat{w}\cdot\partial_i\hat{w})}_{= \frac{1}{2}\partial_i(\hat{w}\cdot\hat{w}) = \frac{1}{2}\partial_i(1) = 0} \hat{w} = 0$$

$$v\, \partial_i \hat{w} - v\, \partial_i \hat{w} = 0 \qquad \checkmark$$

(3)

- scalar product $\langle A, B\rangle = 2\,\mathrm{Tr}(AB)$

We break the symmetry with the vacuum expectation value $\langle \Phi \rangle \xrightarrow{r\to\infty} v\hat{w} = v\,\hat{w}^a T^a$.

$\to$ The unbroken SU(2) direction is $Q = \hat{w}^a T^a$

- $$F_{\mu\nu} \xrightarrow{r\to\infty} \langle Q, G_{\mu\nu}\rangle = 2\,\mathrm{Tr}(Q\, G_{\mu\nu})$$
$$= 2\, \hat{w}^a G^b_{\mu\nu}\, \mathrm{Tr}(T^a T^b)$$
$$= \hat{w}^a G^a_{\mu\nu} = \hat{w} \cdot \vec{G}_{\mu\nu}$$

- $$F_{ij} \xrightarrow{r\to\infty} \hat{w}\cdot(\partial_i \vec{W}_j - \partial_j \vec{W}_i + g\, \vec{W}_i \times \vec{W}_j)$$

$$B_k = -\frac{1}{2} \varepsilon_{kij} F_{ij}$$
$$\xrightarrow{r\to\infty} -\varepsilon_{kij}\, \hat{w}\cdot\partial_i \vec{W}_j - \frac{1}{2} g\, \varepsilon_{kij}\, \hat{w}\cdot(\vec{W}_i \times \vec{W}_j)$$

$$B_k = -\frac{1}{2}\varepsilon_{kij} F_{ij}$$

$$\xrightarrow{r\to\infty} -\varepsilon_{kij}\,\hat w\cdot\partial_i \vec W_j - \frac12 g\,\varepsilon_{kij}\,\hat w\cdot(\vec W_i\times\vec W_j)$$

$$= -\varepsilon_{kij}\,\hat w\cdot\Big(\frac1g\cancel{(\overbrace{\partial_i\partial_j}^{sym}\hat w\times\hat w)} + \frac1g(\partial_j\hat w\times\partial_i\hat w) + (\partial_i c_j)\,\hat w + c_j\,\partial_i\hat w\Big)$$

$$-\frac12 g\,\varepsilon_{kij}\,\hat w\cdot\Big(\frac1{g^2}(\partial_i\hat w\times\hat w)\times(\partial_j\hat w\times\hat w) + \cancel{\overbrace{c_ic_j}^{sym}\,\hat w\cdot\hat w} + \frac1g\underbrace{(\partial_i\hat w\times\hat w)\cdot\hat w}_{=0}\,c_j + \frac1g\underbrace{(\partial_j\hat w\times\hat w)\cdot\hat w}_{=0}\,c_i\Big)$$

$$= -\frac1g\,\varepsilon_{kij}\,\underbrace{\hat w\cdot(\partial_j\hat w\times\partial_i\hat w)}_{=n\,\varepsilon_{jim}\frac{r^m}{r^3}\ \text{(Problem 2)}} - \varepsilon_{kij}\,\partial_i c_j - \varepsilon_{kij}\,c_j\,\underbrace{\hat w\cdot\partial_i\hat w}_{=0}$$

$$-\frac12\,\frac1g\,\varepsilon_{kij}\,(\partial_j\hat w\times\hat w)\cdot\big(\hat w\times(\partial_i\hat w\times\hat w)\big)$$

$$= +\frac ng\,\varepsilon_{kij}\,\varepsilon_{mij}\,\frac{r^m}{r^3} - \varepsilon_{kij}\,\partial_i c_j$$

$$-\frac12\,\frac1g\,\varepsilon_{kij}\,(\partial_j\hat w\times\hat w)\cdot\big(\partial_i\hat w - \hat w\,\underbrace{(\hat w\cdot\partial_i\hat w)}_{=0}\big)$$

$$= \frac ng\cdot 2\,\delta_{km}\,\frac{r^m}{r^3} - \varepsilon_{kij}\,\partial_i c_j - \frac12\,\frac ng\,\varepsilon_{kij}\,\varepsilon_{ijm}\,\frac{r^m}{r^3}$$

$$= \frac ng\,\frac{r^k}{r^3} - \varepsilon_{kij}\,\partial_i c_j$$

(4)

- $r_i\vec W_i\cdot\hat w = \frac1g\, r_i\,(\partial_i\hat w\times\hat w)\cdot\hat w + r_ic_i\,\hat w\cdot\hat w = r_ic_i \overset{!}{=} 0$

- gauge transformation:

$$W_\mu^a \longmapsto W_\mu^a - \varepsilon_{abc}\,\Lambda^b W_\mu^c + \frac1g\,\partial_\mu\Lambda^a$$

$$\vec W_\mu \longmapsto \vec W_\mu - \vec\Lambda\times\vec W_\mu + \frac1g\,\partial_\mu\vec\Lambda$$

- $r_i \vec{W}_i \cdot \hat{w} \longmapsto r_i \vec{W}_i \cdot \hat{w} - r_i (\vec{\Lambda} \times \vec{W}_i) \cdot \hat{w} + \frac{1}{g} r_i \partial_i \vec{\Lambda} \cdot \hat{w} \stackrel{!}{=} 0$

First of all, this gauge transformation shouldn't change the direction $\vec{\Phi} \sim \hat{w}$.

Since $\Phi^a \longmapsto \Phi^a - \varepsilon_{abc} \varphi^b \Phi^c$

$\vec{\Phi} \longmapsto \vec{\Phi} - \vec{\Lambda} \times \vec{\Phi}$

we have to demand $\vec{\Lambda} \parallel \vec{\Phi} \parallel \hat{w}$.

Therefore, $(\vec{\Lambda} \times \vec{W}_i) \cdot \hat{w} = (\hat{w} \times \vec{\Lambda}) \cdot \vec{W}_i = 0$

- $\vec{W}_i \cdot \hat{w} = -\frac{1}{g} \partial_i \vec{\Lambda} \cdot \hat{w} \rightarrow c_i = -\frac{1}{g} \partial_i \vec{\Lambda} \cdot \hat{w}$

$\rightarrow$ There is a possible gauge transformation to get rid of c_i, i.e. we can set $c_i = 0$

(5)

- $\vec{B} = \frac{n}{g} \frac{\vec{r}}{r^3}$

magnetic field of point charge $\vec{B} = \frac{q_m}{4\pi} \frac{\vec{r}}{r^3}$

$\rightarrow$ the magnetic charge is $q_m = \frac{4\pi n}{g} \stackrel{n=1}{=} \frac{4\pi}{g}$