Topological Defects

Problem Sheet 7

4 December 2023

1. Gravitational Field of a String

On a previous problem sheet, we already analyzed the gravitational field of a domain wall. In a similar manner, we want to analyze the gravitational field of a string.

1. *(optional)* Show that in the weak-field approximation the Einstein equation becomes

$$\Box h_{\mu\nu} = -16\pi G \left(T_{\mu\nu} - \frac{1}{2} T \eta_{\mu\nu} \right). \tag{1}$$

2. Take the theory

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* (D^\mu \phi) - V(\phi) \tag{2}$$

and consider a string located along the z-axis. Calculate the components T_{00}, T_{33}, T_{0i}, and T_{3i}.

3. To calculate the other components, we will use the approximation

$$\tilde{T}_{\mu\nu}(x, y, z) = \delta(x)\delta(y) \int T_{\mu\nu}(x, y) \mathrm{d}x \mathrm{d}y. \tag{3}$$

Show that the energy-momentum tensor takes the form

$$\tilde{T}_{\mu\nu}(x, y, z) = \varepsilon(x, y, z)\delta(x)\delta(y) \mathrm{diag}(1, 0, 0, -1), \tag{4}$$

where $\varepsilon(x, y, z)$ is the energy density of the string.

4. Insert this into equation (1) and find the solution for $h_{\mu\nu}$. You will find that the non-zero components are h_{11} and h_{22}.
Hint : Solve the equation in cylindrical coordinates.

The metric can be written as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}z^2 - \mathrm{d}r'^2 - r'^2 \mathrm{d}\varphi'^2, \tag{5}$$

where r' and ϕ' depend on r and ϕ. This means that in the (t, z, r', φ') coordinates, the trajectories are straight lines, but translated into the observable coordinates (t, z, r, φ) the trajectories are curved.

5. Make a sketch of a light beam that passes by the string.

2. Non-Abelian Gauge Theory

Let us consider the following Lagrangian

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\,(G_{\mu\nu}G^{\mu\nu}) + \mathrm{Tr}\,\big((D_\mu\phi)^\dagger(D^\mu\phi)\big)\,. \tag{6}$$

The scalar field ϕ is given in the adjoint representation of $SU(N)$ with $\phi = \phi^a T^a$, where T^a are the generators of the $SU(N)$ group normalized by $\mathrm{Tr}\,T^aT^b = \delta^{ab}/2$. Remember that the generators satisfy the algebra

$$[T^a, T^b] = if^{abc}T^c. \tag{7}$$

The covariant derivative is given by

$$D_\mu\phi = \partial_\mu\phi - ig[W_\mu, \phi], \tag{8}$$

and the field strength tensor is

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig[W_\mu, W_\nu], \tag{9}$$

with $W_\mu = W_\mu^a T^a$.

1. The gauge transformation of ϕ is given by

$$\phi \mapsto U\phi U^\dagger, \tag{10}$$

   with $U = e^{iT^a\varphi^a(x)}$. How does the gauge field W_μ transform, if the covariant derivative transforms in the same way as the field ϕ ?
2. Show that the Lagrangian is invariant under $SU(N)$ gauge transformation.
3. How does the infinitesimal gauge transformation look like for the components ϕ^a and W_μ^a ?
4. Find the equations of motion for both, the gauge field and the scalar field.
5. If φ^a is a constant, what is the conserved Noether current ?
6. Prove the Bianchi identity

$$\varepsilon^{\mu\nu\alpha\beta}D_\nu G_{\alpha\beta} = 0. \tag{11}$$

   Hint : Use the Jacobi identity.
7. Assuming W_μ is a pure gauge, i.e. $W_\mu = \frac{i}{g}U\partial_\mu U^\dagger$, show that $\mathrm{Tr}\,(G_{\mu\nu}G^{\mu\nu}) = 0$.