Topological Defects

Problem Sheet 6 27 November 2023

1. Non-Static Solutions in the Sine-Gordon Model

In this problem, we will analyze some special solutions of the so-called Sine-Gordon model, given by the Lagrangian (in $1+1$ dimensions)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^4}{\lambda}\left(1-\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right)\right). \tag{1}$$

This model allows static kink solutions of mass $M_{DW} = \frac{8m^3}{\lambda}$. But in this problem, we will focus on non-static configurations. Since the calculations are tedious and hard, feel free to use the computer.

1. What are the mass units of λ and m ? What is the mass of ϕ ?
2. Convince yourself that
$$\phi = \frac{4m}{\sqrt{\lambda}}\arctan\left(u\frac{\sinh(\gamma m x)}{\cosh(\gamma m u t)}\right) \tag{2}$$
is a solution of the field equations. Here $\gamma = 1/\sqrt{1-u^2}$
3. What is described by this solution ?
4. Show that the energy of this configuration is
$$E = 2M_{DW}\gamma. \tag{3}$$
5. Another solution of the field equations is
$$\phi = \frac{4m}{\sqrt{\lambda}}\arctan\left(\frac{1}{u}\frac{\sinh(\gamma m u t)}{\cosh(\gamma m x)}\right) \tag{4}$$
What is described by this ?
6. Take the latter solution and replace u by is. What does the new solution describe ?

2. The Moduli Approximation

The goal of this problem is to introduce the moduli approximation. The so-called moduli space is described by the collective coordinates, which are parameters that don't change the total energy for the system like for example the position of a soliton. As an example, let us start with the kink solution of a real scalar field theory in $1+1$ dimensions. Here, the only collective coordinate is the position of the kink a. This position can change when we give the kink an initial momentum. We can describe this moving kink by $\phi(x-a(t))$, where $a(t)$ is changing slowly.

1. Show that the kinetic energy can be written as

$$T = \frac{1}{2} M \dot{a}^2, \tag{5}$$

where M is the total mass of the kink.

Now let us switch to the vortex case in $2+1$ dimensions. The situation becomes non-trivial because the transformation $x_i \to x_i - a_i(t)$ requires an additional gauge transformation.

2. Show that the Gauss constraint (zero component of the field equation for A_μ) becomes violated, when we transform the field by

$$\phi(x) \mapsto \phi(x) - a_i(t)\partial_i \phi(x), \tag{6}$$

$$A_i(x) \mapsto A_i(x) - a_j(t)\partial_j A_i(x), \tag{7}$$

where now $a_i(t)$ is very small.

Note that with this transformation we did not change A_0, which is zero for the vortex solution. To have a non-violated Gauss constraint, A_0 has to transform when we apply the transformation $x_i \to x_i - a_i(t)$. The Gauss constraint will require the transformation $A_0(x) \mapsto -\dot{a}(t)A_i(x)$. With an additional gauge transformation, we can set A_0 back to zero.

3. With an additional small gauge transformation, the transformation becomes

$$\phi(x) \mapsto \phi(x) + i\alpha(x)\phi(x) - a_i(t)\partial_i \phi(x), \tag{8}$$

$$A_i(x) \mapsto A_i(x) - \frac{1}{g}\partial_i \alpha(x) - a_j(t)\partial_j A_i(x). \tag{9}$$

Show that for $\alpha(x) = -gA_i(x)a_i(t)$ the Gauss constraint is invariant under the above transformation.

4. Show that the kinetic energy of a slow-moving vortex is

$$T = \int \mathrm{d}^2 x \left(\frac{1}{2} F_{ik}F_{jk} + (D_i\phi)^*(D_j\phi) \right) \dot{a}_i \dot{a}_j. \tag{10}$$

5. In the BPS limit, the kinetic energy takes the form

$$T = \frac{1}{2} M_{\text{vortex}} g_{ij} \dot{a}_i \dot{a}_j, \tag{11}$$

where g_{ij} is a diagonal matrix. Which values have the components g_{ij}?
Hint : Remember that the vortex in the BPS limit satisfies $(D_1 + iD_2)\phi = 0$.

g_{ij} can be interpreted as the metric on the moduli space - the space of the collective coordinates a_i. Since there is only a kinetic term for the collective coordinates, the trajectories of the vortex described by a_i are given by the geodesics.

6. In the case of a single vortex, which geometric form has the moduli space and what are the allowed trajectories of the vortex?

3. Vortex Scattering

In this problem, we will analyze the scattering of multiple vortices qualitatively. We will start with two vortices that collide head-on.

1. The geometric form of the moduli space for a vortex-vortex configuration is

$$\mathcal{M} = \mathbb{C} \times \frac{\mathbb{C}}{Z_2}. \tag{12}$$

Write down all the collective coordinates. Which coordinates correspond to $\mathbb{C}$ and which coordinates correspond to $\mathbb{C}/Z_2$. $\mathbb{C}/Z_2$ can be imagined as a cone. Can you explain why ?
Hint : First, note that $\mathbb{C} \cong \mathbb{R}^2$. In addition, $\mathbb{C}/Z_2$ means that only half of the two-dimensional space is considered.

The geometric form (12) is just an approximation for long distances. When the two vortices come closer to each other, interactions between them play a non-negligible role. A more detailed analysis of the moduli space reveals that the cone corresponding to $\mathbb{C}/Z_2$ has a rounded tip.

2. Draw the cone and the trajectory on it for a head-on collision of the two vortices.
3. Draw the trajectory of the two vortices in the coordinate space.
4. Draw the scalar field ϕ in a vector plot before and after the collision.

Now let us proceed one step further. Instead of two vortices, we will take four vortices.

5. How many collective coordinates are there for such configurations ?

We will constrain some collective coordinates. Let us consider four vortices, each of which is located at a corner of a square. We will analyze only the moduli subspace for which this condition holds.

6. How many collective coordinates has this subspace ? Can you guess the geometric form of this moduli subspace ?
7. How does the trajectory of the four vortices look like in a scattering process ?