Problem 1: (have a look at Shifman - Advanced Topics in QFT)

- $S = \int d^2x \left[\frac{1}{2} (\partial_\mu \phi)^2 - \frac{\lambda}{4} (\phi^2 - v^2)^2 \right]$

decomposition: $\phi(t,z) = \phi_k(z) + \chi(t,z)$

kink in background ↑ ↑ fluctuations

(1)

- $$S[\phi] = \int d^2x \left[\frac{1}{2} (\partial_\mu \phi_k + \partial_\mu \chi)^2 - V(\phi_k + \chi) \right]$$

$$= \int d^2x \left[\underline{\frac{1}{2} (\partial_\mu \phi_k)^2} + (\partial_\mu \phi_k)(\partial^\mu \chi) + \frac{1}{2} (\partial_\mu \chi)^2 \underline{- V(\phi_k)} - V'(\phi_k)\chi - \frac{1}{2} V''(\phi_k)\chi^2 + \dots \right]$$

integration by parts

$$\overset{\downarrow}{=} S[\phi_k] + \int d^2x \Big[\underbrace{- (\partial_\mu \partial^\mu \phi)\chi - V'(\phi_k)\chi}_{=0 \text{ by e.o.m.}} + \frac{1}{2} \dot{\chi}^2 + \frac{1}{2} \chi \partial_z^2 \chi - \frac{1}{2} V''(\phi_k)\chi^2 + \dots \Big]$$

$$= S[\phi_k] + \int d^2x \left[\frac{1}{2} \dot{\chi}^2 - \frac{1}{2} \chi L_2 \chi \right]$$

where $L_2 = -\partial_z^2 + V''(\phi_k)$

$$= -\partial_z^2 + m^2 \left[1 - \frac{3}{2} \left(\cosh \frac{mz}{2} \right)^{-2} \right]$$

(2)

- $L_2 \chi_n(z) = \omega_n^2 \chi_n(z)$

$\int dz\, \chi_n(z) \chi_m(z) = \delta_{mn}$

$\chi(t,z) = \sum_n a_n(t) \chi_n(z)$

- $H = \int dz \left[\frac{1}{2} \dot{\chi}^2 + \frac{1}{2} \chi L_2 \chi \right]$

$$= \int dz \left[\frac{1}{2} \left(\sum_n \dot{a}_n(t) \chi_n(z) \right)^2 + \frac{1}{2} \left(\sum_n a_n(t) \chi_n(z) \right) \left(\sum_m a_m(t) L_2 \chi_m(z) \right) \right]$$

$$= \sum_{m,n} \left[\frac{1}{2} \dot{a}_n \dot{a}_m \int dz\, \chi_n(z) \chi_m(z) + \frac{1}{2} a_n a_m \int dz\, \chi_n(z)\, \omega_m^2\, \chi_m(z) \right]$$

$$= \sum_n \left(\frac{1}{2} \dot{a}_n^2 + \frac{1}{2} a_n^2 \omega_n^2 \right)$$

- This is the Hamiltonian for multiple decoupled harmonic oscillators.

(3)

- zero mode: mode that does not change the energy

  $\phi_k(z - z_0)$ and $\phi_k(z - z_0 - \delta z_0)$ have

  same energy $\rightarrow$ fluctuations proportional

  to $\frac{\partial}{\partial z_0} \phi_k(z - z_0)$ carry no energy

- $$\chi_0(z) = \mathcal{N} \left. \frac{\partial \phi_k(z - z_0)}{\partial z_0} \right|_{z_0 = 0} = \mathcal{N} \cdot \frac{m}{\sqrt{2}\lambda} \frac{1}{\cosh^2\left(\frac{mz}{2}\right)} \quad \left(-\frac{m}{2}\right)$$

  ↑ free choice

$$= -\mathcal{N} \frac{m^2}{2\sqrt{2}\lambda} \frac{1}{\cosh^2\left(\frac{mz}{2}\right)}$$

$$\int dz\, \chi_0(z) \chi_0(z) = \mathcal{N}^2 \int dz \left(\left. \frac{\partial \phi_k(z - z_0)}{\partial z_0} \right|_{z_0 = 0} \right)^2$$

$$= \mathcal{N}^2 \int dz \left(\frac{\partial \phi_k(z)}{\partial z} \right)^2 \qquad \text{with Bogomolny eq.}$$

$$= \mathcal{N}^2 \int dz \left[\frac{1}{2} \left(\frac{\partial \phi_k}{\partial z} \right)^2 + V(\phi_k) \right]$$

$$= \mathcal{N}^2 M_k \overset{!}{=} 1$$

↑ kink mass

$$\rightarrow \mathcal{N} = \pm \frac{1}{\sqrt{M_k}}$$

$$\rightarrow N = \pm \frac{1}{\sqrt{M_k}}$$

$\rightarrow$ the normalized zero mode solution is

$$\chi_0(z) = \frac{1}{\sqrt{M_k}} \frac{m^2}{2\sqrt{2}\lambda} \frac{1}{\cosh^2(\frac{mz}{2})} = \sqrt{\frac{3m}{8}} \frac{1}{\cosh^2(\frac{mz}{2})}$$

(4)

- $L_2 \chi_0$

$$= \left[-\partial_z^2 + m^2\left(1 - \frac{3}{2}\left(\cosh \frac{mz}{2}\right)^{-2}\right)\right] \sqrt{\frac{3m}{8}} \frac{1}{\cosh^2(\frac{mz}{2})}$$

$$= \sqrt{\frac{3m}{8}} \cdot m \cdot \partial_z \left(\frac{\sinh(\frac{mz}{2})}{\cosh^3(\frac{mz}{2})}\right) + m^2 \sqrt{\frac{3m}{8}} \left(\frac{1}{\cosh^2(\frac{mz}{2})} - \frac{3}{2} \frac{1}{\cosh^4(\frac{mz}{2})}\right)$$

$$= m\sqrt{\frac{3m}{8}} \left(\frac{m}{2} \frac{1}{\cosh^2(\frac{mz}{2})} - 3\,\frac{m}{2} \frac{\sinh^2(\frac{mz}{2})}{\cosh^4(\frac{mz}{2})}\right) + m^2\sqrt{\frac{3m}{8}} \left(\frac{1}{\cosh^2(\frac{mz}{2})} - \frac{3}{2} \frac{1}{\cosh^4(\frac{mz}{2})}\right)$$

$$= m^2\sqrt{\frac{3m}{8}} \left[\frac{1/2}{\cosh^2(\frac{mz}{2})} - \frac{3/2}{\cosh^2(\frac{mz}{2})} + \frac{3/2}{\cosh^4(\frac{mz}{2})} + \frac{1}{\cosh^2(\frac{mz}{2})} - \frac{3/2}{\cosh^4(\frac{mz}{2})}\right]$$

$$= 0 \quad \rightarrow \quad \omega_0 = 0$$

- This confirms that the zero carries no energy

(5)

- $L_2 = -\partial_z^2 + V''$

Idea: If it is possible to write $L_2 = P^+P$,

all eigenvalues are positive:

$$L_2 \chi_n = \omega_n^2 \chi_n$$

$$\chi_n P^+ P \chi_n = \chi_n \omega_n^2 \chi_n$$

$$\rightarrow \underbrace{|P\chi_n|^2}_{\geq 0} = \omega_n^2 \underbrace{\chi_n^2}_{\geq 0} \quad \Rightarrow \quad \omega_n \geq 0$$

- It remains to prove that $L_2 = P^+P$.

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Ansatz: $P = \partial_z + f(\phi_k)$

$$P\chi_0 = \partial_z \chi_0 + f(\phi_k)\,\chi_0$$
$$= \sqrt{\frac{3m}{8}}\,(-2)\,\frac{m}{2}\,\frac{\sinh\left(\frac{mz}{2}\right)}{\cosh^3\left(\frac{mz}{2}\right)} + f(\phi_k)\sqrt{\frac{3m}{8}}\,\frac{1}{\cosh^2\left(\frac{mz}{2}\right)}$$
$$= 0 \quad \text{for} \quad f(\phi_k) = m \tanh\left(\frac{mz}{2}\right)$$

This choice would already give $L_2 \chi_0 = 0$

If we calculate P^+P, let's see if we get L_2:

$$P^+P = \left(-\partial_z + m\tanh\left(\frac{mz}{2}\right)\right)\left(\partial_z + m\tanh\left(\frac{mz}{2}\right)\right)$$
$$= -\partial_z^2 - \frac{m^2}{2}\,\frac{1}{\cosh^2\left(\frac{mz}{2}\right)} - m\tanh\left(\frac{mz}{2}\right)\partial_z + m\tanh\left(\frac{mz}{2}\right)\partial_z$$
$$+ m^2\left(1 - \frac{1}{\cosh^2\left(\frac{mz}{2}\right)}\right)$$
$$= -\partial_z^2 + m^2\left(1 - \frac{3}{2}\,\frac{1}{\cosh^2\left(\frac{mz}{2}\right)}\right) = L_2$$

Why $\partial_z^+ = -\partial_z$?

$$\langle f \,|\, \partial_z g \rangle = \langle \partial_z^+ f \,|\, g \rangle$$
$$\int dz\; f(z)\,(\partial_z g(z)) \underset{\text{partial int.}}{=} \int dz\; (-\partial_z f(z))\, g(z)$$

(G)

- $$S[\phi_k] = \int dt\,dz \left[\frac{1}{2}\dot{\phi}_k^2 - \frac{1}{2}\phi_k'^2 - V(\phi_k)\right]$$
$$= \int dt\,dz \left[\frac{1}{2}\phi_k'^2\,\dot{z}_0^2 - \frac{1}{2}\phi_k'^2 - V(\phi_k)\right]$$

$$= \int dt \left[\frac{1}{2} M_k \dot{z}_0^2 - M_k \right]$$

$$\rightarrow H_0 = M_k + \frac{M_k}{2} \dot{z}_0^2$$

$$\rightarrow H = M_k + \frac{M_k}{2} \dot{z}_0^2 + \sum_{n \neq 0} \left[\frac{1}{2} \dot{a}_n^2 + \frac{\omega_n^2}{2} a_n^2 \right]$$

(7)

- We want to solve the equation

$$L_2 \Psi_2 = \left(-\partial_z^2 + m^2 \left(1 - \frac{3}{2} \frac{1}{\cosh^2\left(\frac{mz}{2}\right)} \right) \right) \Psi_2 = \omega^2 \Psi_2$$

We know that we can write

$$L_2 = P^+ P \quad \text{with} \quad P = \partial_z + m \cdot \tanh\left(\frac{mz}{2}\right)$$

$$P^+ = -\partial_z + m \cdot \tanh\left(\frac{mz}{2}\right)$$

- Let us define $L_1 := P P^+$.

Since $P \underbrace{L_2 \Psi_2}_{\lambda \Psi_2} = L_1 P \Psi_2 = \lambda P \Psi_2$, L_2 and L_1 share the same eigenvalues except for the ground state, because $P \Psi_2 = 0$.

If Ψ_1 is an eigenstate of L_1, we can obtain the corresponding eigenstate of L_2 (with the same eigenvalue) by $\Psi_2 = P^+ \Psi_1$, because

$$L_1 \Psi_1 = \lambda \Psi_1 \rightarrow P^+ \underbrace{L_1 \Psi_1}_{= \lambda \Psi_1} = L_2 P^+ \Psi_1 := \lambda P^+ \Psi^1$$

$$\rightarrow L_2 \Psi_2 = \lambda \Psi_2$$

- $L_1 = P P^+$

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$$= \left(\partial_z + m \cdot \tanh\left(\frac{mz}{2}\right)\right)\left(-\partial_z + m \cdot \tanh\left(\frac{mz}{2}\right)\right)$$

$$= -\partial_z^2 + m^2 \left(1 - \frac{1}{2} \frac{1}{\cosh^2\left(\frac{mz}{2}\right)}\right)$$

A solution to $L_1 \Psi_1 = \omega^2 \Psi_1$ is $\Psi_1 \sim \frac{1}{\cosh\left(\frac{mz}{2}\right)}$

with $\omega^2 = \frac{3m^2}{4}$

$\rightarrow \Psi_2 = P^+ \Psi_1 \sim \frac{\sinh\left(\frac{mz}{2}\right)}{\cosh^2\left(\frac{mz}{2}\right)}$ with the same eigenvalue ω^2

- We found the second mode. Let us continue with the same procedure.

$L_1 \Psi_1 = \omega^2 \Psi_1$

$\rightarrow \tilde{L}_1 \Psi_1 = 0$ with $\tilde{L}_1 = L_1 - \omega^2$

Let us try to get the decomposition

$\tilde{L}_1 = \tilde{P}^+ \tilde{P}$

$\tilde{P} = \partial_z + f(z)$

$$\tilde{P} \Psi_1 = \left(\partial_z + f(z)\right) \frac{1}{\cosh\left(\frac{mz}{2}\right)}$$

$$= -\frac{m}{2} \frac{\sinh\left(\frac{mz}{2}\right)}{\cosh^2\left(\frac{mz}{2}\right)} + f(z) \frac{1}{\cosh\left(\frac{mz}{2}\right)} = 0$$

$$\rightarrow \tilde{P} = \partial_z + \frac{m}{2} \tanh\left(\frac{mz}{2}\right)$$

$$\tilde{P}^+ \tilde{P} = \left(-\partial_z + \frac{m}{2} \tanh\left(\frac{mz}{2}\right)\right)\left(\partial_z + \frac{m}{2} \tanh\left(\frac{mz}{2}\right)\right)$$

$$= -\partial_z^2 - \frac{m^2}{4}\frac{1}{\cosh^2\left(\frac{mz}{2}\right)} + \frac{m^2}{4}\left(1 - \frac{1}{\cosh^2\left(\frac{mz}{2}\right)}\right)$$

$$= -\partial_z^2 - \frac{m^2}{2}\frac{1}{\cosh^2\left(\frac{mz}{2}\right)} + \frac{m^2}{4} = \tilde{L}_1$$

$\rightarrow$ We found a working decomposition

- In the same way as before we calculate $L_0 := \tilde{P}\tilde{P}^+$, because L_0 and $\tilde{L}_1$ share the same eigenvalues.

$$L_0 = \tilde{P}\tilde{P}^+ = \left(\partial_z + \frac{m}{2}\tanh\left(\frac{mz}{2}\right)\right)\left(-\partial_z + \frac{m}{2}\tanh\left(\frac{mz}{2}\right)\right)$$

$$= -\partial_z^2 + \frac{m^2}{4}\frac{1}{\cosh^2\left(\frac{mz}{2}\right)} + \frac{m^2}{4}\frac{\sinh^2\left(\frac{mz}{2}\right)}{\cosh^2\left(\frac{mz}{2}\right)}$$

$$= -\partial_z^2 + \frac{m^2}{4}$$

$L_0 \Psi_0 = \tilde{\omega}^2 \Psi_0$

solution: $\Psi_0 \sim e^{ipz}$ with $\tilde{\omega}^2 = p^2 + \frac{m^2}{4}$

We obtained a continuous spectrum.

The eigenvalues of $\tilde{L}_1$ are $\tilde{\omega}^2 = p^2 + \frac{m^2}{4}$

and thus the corresponding eigenvalues of $L_1 = \tilde{L}_1 + \omega^2$

are $\omega^2_{contin.} = \frac{m^2}{4} + \frac{3}{4}m^2 + p^2 = m^2 + p^2$

and also the eigenvalues of L_2 are

$$\omega^2_{contin.} = m^2 + p^2$$

- Now we can calculate the corresponding eigenstates:

$\Psi_2 = P^+\Psi_1 = P^+\tilde{P}^+\Psi_0$

$$\sim \left(-\partial_z + m\tanh\left(\frac{mz}{2}\right)\right)\left(-\partial_z + \frac{m}{2}\tanh\left(\frac{mz}{2}\right)\right)e^{ipz}$$

$$\sim \left(-\partial_z + m\tanh\left(\frac{mz}{2}\right)\right)\left(-\partial_z + \frac{m}{2}\tanh\left(\frac{mz}{2}\right)\right) e^{ipz}$$

$$= \left(-\partial_z + m\tanh\left(\frac{mz}{2}\right)\right)\left(-ip + \frac{m}{2}\tanh\left(\frac{mz}{2}\right)\right) e^{ipz}$$

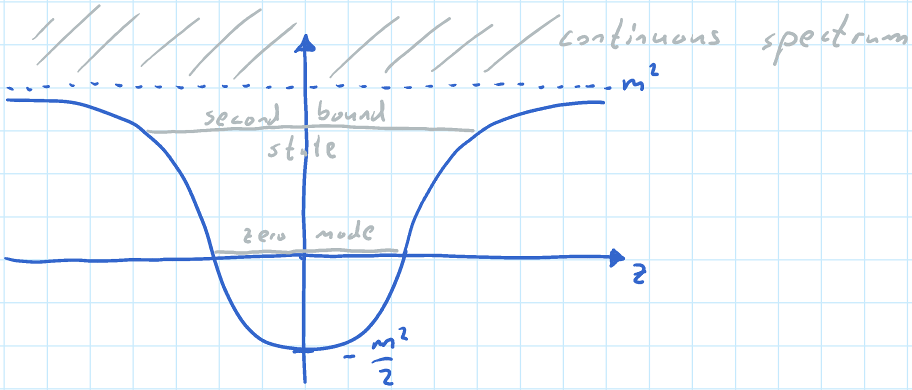
$$= \left[\left(-ip + \frac{m}{2}\tanh\left(\frac{mz}{2}\right)\right)(-ip) - \frac{m^2}{4}\frac{1}{\cosh^2\left(\frac{mz}{2}\right)}\right.$$

$$\left. - ipm\tanh\left(\frac{mz}{2}\right) + \frac{m^2}{2}\tanh^2\left(\frac{mz}{2}\right)\right] e^{ipz}$$

$$= \left(-p^2 - \frac{3m}{2} ip\tanh\left(\frac{mz}{2}\right) + \frac{m^2}{2} - \frac{3m^2}{4}\frac{1}{\cosh^2\left(\frac{mz}{2}\right)}\right) e^{ipz}$$

- Sketch:

$$\left[-\partial_z^2 + \underbrace{m^2\left(1 - \frac{3}{2}\left(\cosh\frac{mz}{2}\right)^{-2}\right)}_{\text{Like potential in Schrödinger eq.}}\right]\chi_n(z) = \omega_n^2\,\chi_n(z)$$



(8)

- At $z \to -\infty$ the is an incoming wave e^{ipz}.

Parts of it can transmit and parts of it can reflect:

transmission coefficient

$$\lim_{z\to\infty} \varphi(z) = T(p)\, e^{ipz}$$

$$\lim_{z\to-\infty} \varphi(z) = e^{ipz} + R(p)\, e^{-ipz}$$

reflection coefficient

- $$\lim_{z\to\infty} \chi_p(z) = \left(-p^2 - \frac{3m}{2} ip + \frac{m^2}{2}\right) e^{ipz}$$

$$\lim_{z\to-\infty} \chi_p(z) = \left(-p^2 + \frac{3m}{2} ip + \frac{m^2}{2}\right) e^{ipz}$$

$$\to T(p) = \frac{-p^2 - \frac{3m}{2} ip + \frac{m^2}{2}}{-p^2 + \frac{3m}{2} ip + \frac{m^2}{2}} \quad , \quad R(p) = 0 \quad \text{(reflectionless potential)}$$

$$= \left(\frac{m+ip}{m-ip}\right)\left(\frac{m+2ip}{m-2ip}\right)$$

- phase shift:

$$\chi_p(z\to-\infty) = e^{ipz}$$

$$\chi_p(z\to+\infty) = T(p)\, e^{ipz} = e^{ipz+i\varphi}$$

$$\to e^{i\varphi} = T(p) = \left(\frac{m+ip}{m-ip}\right)\left(\frac{m+2ip}{m-2ip}\right)$$

(g)

- boundary conditions:

$$A\,\chi_p\left(\frac{L}{2}\right) + B\,\chi_p\left(-\frac{L}{2}\right) = 0$$

$$A\,\chi_p\left(-\frac{L}{2}\right) + B\,\chi_p\left(\frac{L}{2}\right) = 0$$

We have a non-trivial solution if

$$\chi_p\left(\frac{L}{2}\right) = \pm\, \chi_p\left(-\frac{L}{2}\right)$$

$$e^{\frac{ipL}{2} + i\varphi} = \pm\, e^{-\frac{ipL}{2}}$$

$$e^{ipL + i\varphi} = \pm 1$$

$$\to pL + \varphi = \pi n \quad , \quad n = 0, 1, \ldots$$

$$\to \omega_n = \sqrt{m^2 + p_n^2} \quad \text{with} \quad p_n = \frac{\pi n - \varphi}{L}$$

$$\rightarrow \omega_n = \sqrt{m^2 + p_n^2} \quad \text{with} \quad p_n = \frac{\pi n - \varphi}{L}$$

- The vacuum energy is given by all free waves without the kink in the background: $p_n^0 = \frac{\pi n}{L}$

$$E_{vac} = \frac{1}{2} \sum_{n=0}^{\infty} \sqrt{m^2 + (p_n^0)^2}$$

We need to subtract this energy from the quantum corrected mass of the kink:

$$M^{quant} = M + \frac{\omega_1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \sqrt{m^2 + p_n^2} - E_{vac}$$

- ω_1 can be calculated by $L\chi_1 = \omega_1^2 \chi_1$ and gives $\omega_1^2 = \frac{3m^2}{4}$.

With $p_n = p_n^0 - \frac{\varphi}{L}$:

$$M^{quant} = M + \frac{\sqrt{3}m}{4} + \frac{1}{2} \sum_{n=0} \left[\sqrt{m^2 + (p_n^0)^2 - \frac{2\varphi}{L} p_n^0 + \cancel{\frac{\varphi^2}{L^2}}} - \sqrt{m^2 + (p_n^0)^2} \right]$$

for large L

Taylor

$$\approx M + \frac{\sqrt{3}m}{4} + \frac{1}{2} \sum_{n=0} \left[\sqrt{m^2 + (p_n^0)^2} - \frac{\frac{\varphi}{L} p_n^0}{\sqrt{m^2 + (p_n^0)^2}} - \sqrt{m^2 + (p_n^0)^2} \right.$$

$$= M + \frac{\sqrt{3}m}{4} - \frac{1}{2} \sum_{n=0} \frac{\frac{\varphi}{L} p_n^0}{\sqrt{m^2 + (p_n^0)^2}}$$

(10)

- continuous Limit: $\sum_n = \sum_{\frac{p_n^0 L}{\pi}} \longrightarrow \int_0^{\infty} \frac{dp\,L}{\pi}$

$$M^{quant} \longrightarrow M + \frac{\sqrt{3}\,m}{4} - \frac{1}{2\pi}\int_0^\infty dp \, \frac{\varphi p}{\sqrt{m^2+p^2}}$$

$$\left[\frac{d}{dp}\sqrt{m^2+p^2} = \frac{p}{\sqrt{m^2+p^2}} \right]$$

partial int.

$$\overset{\text{partial int.}}{=} M + \frac{\sqrt{3}\,m}{4} + \frac{1}{2\pi}\int_0^\infty dp \, \frac{d\varphi}{dp}\sqrt{m^2+p^2}$$

$$\varphi = -i \ln\left(\left(\frac{m+ip}{m-ip}\right)\left(\frac{m+2ip}{m-2ip}\right)\right)$$

$$\frac{d\varphi}{dp} = -i\left(\frac{m-ip}{m+ip}\right)\left(\frac{m-2ip}{m+2ip}\right)\left[\left(\frac{(m-ip)i-(m+ip)(-i)}{(m-ip)^2}\right)\left(\frac{m+2ip}{m-2ip}\right) + \left(\frac{m+ip}{m-ip}\right)\left(\frac{(m-2ip)2i-(m+2ip)(-2i)}{(m-2ip)^2}\right)\right]$$

$$= -i\,\frac{2im}{m^2+p^2} - i\,\frac{4im}{m^2+4p^2}$$

$$= \frac{2}{m}\left(\frac{1}{1+y^2} + \frac{2}{1+4y^2}\right) \quad \text{where} \quad y = \frac{p}{m}$$

$$\rightarrow M^{quant} \approx M + \frac{\sqrt{3}\,m}{4} + \frac{m}{\pi}\int_0^\infty dy \left(\frac{1}{1+y^2} + \frac{2}{1+4y^2}\right)\sqrt{1+y^2}$$

(11)

- Since the integral is logarithmically divergent we can approximate the integral (only large y contribute significantly)

$$\frac{m}{\pi}\int_0^\infty dy\left(\frac{1}{\cancel{1}+y^2} + \frac{2}{\cancel{1}+4y^2}\right)\sqrt{\cancel{1}+y^2}$$

$$\approx \frac{m}{\pi}\int_0^\infty dy \, \frac{3}{2y^2}\cdot y = \frac{3m}{2\pi}\int_0^\infty dy \, \frac{1}{y}$$

- Introduce cutoff $p = M_{UV}$

$$\frac{3m}{2\pi}\int_0^{\frac{M_{UV}}{m}} dy \, \frac{1}{y} = \frac{3m}{2\pi}\ln\left(\frac{M_{UV}}{m}\right) - \cancel{\frac{3m}{2\pi}\ln(0)}$$

this is an artificial singularity,

this is an artificial singularity, which comes from the approximation above. We can ignore this, because $\ln(y_{min})$ is way smaller than $\ln(M_{UV}/m)$

$$\Rightarrow M^{quant} \approx M + \frac{3m}{2\pi} \ln\left(\frac{M_{UV}}{m}\right)$$