Topological Defects

Problem Sheet 5 20 November 2023

1. Quantization of the Domain Wall

Let us consider the scalar field theory with the action

$$S = \int \mathrm{d}^2x \left[\frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda^2}{4}(\phi^2 - v^2)^2\right]. \tag{1}$$

In the following, we will analyze the decomposition

$$\phi(t, z) = \phi_k(z) + \chi(t, z), \tag{2}$$

where χ are fluctuations around the background described by the kink solution $\phi_k(z) = \frac{m}{\sqrt{2}\lambda}\tanh\left(\frac{m}{2}z\right)$ with mass $M_k = \frac{m^3}{3\lambda^2}$, where $m = \sqrt{2}\lambda v$.

1. Expand the action as

$$S[\phi] = S[\phi_k] + \int \mathrm{d}^2x \left[\frac{1}{2}\dot{\chi}^2 - \frac{1}{2}\chi L_2 \chi + \dots\right] \tag{3}$$

   and determine L_2.

L_2 is a Hermitian operator and thus its eigenfunctions form a complete basis. We will write

$$L_2 \chi_n(z) = \omega_n^2 \chi_n(z), \tag{4}$$

where the orthonormal function $\chi_n(z)$ are normalized by

$$\int \mathrm{d}z \chi_n(z)\chi_m(z) = \delta_{nm}. \tag{5}$$

Now we can write down the decomposition

$$\chi(t, z) = \sum_n a_n(t)\chi_n(z). \tag{6}$$

2. Show that the Hamiltonian corresponding to the fluctuations χ can be written as

$$H = \sum_n \left[\frac{1}{2}\dot{a}_n^2 + \frac{\omega_n^2}{2}a_n^2\right]. \tag{7}$$

   Recall your quantum mechanics course and interpret the result.

The energy of the system is preserved under translations of the domain wall. Therefore, all domain walls $\phi_k(z - z_0)$ have the same energy. That's why a mode proportional to the derivative $\partial_{z_0}\phi_k(z - z_0)$ does not give an energy contribution. We call this mode the *zero mode*.

3. Find the normalized zero mode solution.
4. Show that this mode corresponds to the eigenvalue $\omega_0 = 0$.
5. Show that L_2 has no negative eigenvalues.
   Hint : Try to write $L_2 = P^\dagger P$.
6. Let's assume that for small velocities the dynamics of the kink can be described by $\phi_k(t,z) = \phi_k(z - z_0(t))$. Show that the full Hamiltonian can be written as

$$H = M + \frac{M}{2}\dot{z}_0^2 + \sum_{n\neq 0}\left[\frac{1}{2}\dot{a}_n^2 + \frac{\omega_n^2}{2}a_n^2\right]. \tag{8}$$

Next, we want to calculate the quantum effects on the mass of the domain wall. From the Hamiltonian, we can read of

$$M^{\text{quantum}} = M + \sum_{n\neq 0}\frac{\omega_n}{2}. \tag{9}$$

Equation (4) has two bounded solutions with $\omega^2 < m^2$ and a continous spectrum for $\omega^2 > m^2$. One bounded solution corresponds to the zero mode and the other bounded solution is

$$\chi_1(z) \propto \frac{\sinh\frac{mz}{2}}{\cosh^2\frac{mz}{2}}. \tag{10}$$

One possible solution for the continuous spectrum is given by

$$\chi_p(z) \propto \left(-p^2 - \frac{3m}{2}ip\tanh\left(\frac{mz}{2}\right) + \frac{m^2}{2} - \frac{3m^2}{4}\frac{1}{\cosh^2\left(\frac{mz}{2}\right)}\right)e^{ipz}; \tag{11}$$

where $p^2 = \omega_p^2 - m^2$.

7. *(optional and hard)* Derive the solutions (10) and (11).
8. Far away from the domain wall, equation (11) describes a free wave. During the interaction with the domain wall, it obtains a phase shift. One can denote this by

$$\chi_p(z\to -\infty) = e^{ipz}, \tag{12}$$
$$\chi_p(z\to \infty) = e^{ipz+i\phi}. \tag{13}$$

   Calculate the phase shift $e^{i\phi}$.

To quantize the continuous part, we need to choose a finite volume of length L. At the boundary, we can demand that a general solution $A\chi_p(z) + B\chi_p(-z)$ vanishes there. Furthermore, in quantum physics, we always calculate energies with respect to the vacuum (no domain wall). Therefore, we need to subtract the vacuum energy corresponding to free waves.

9. Show that the quantum corrected mass of the domain wall is given by

$$M^{\text{quant}} \approx M + \frac{\sqrt{3}m}{4} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{\frac{\phi}{L}p_n^0}{\sqrt{m^2 + (p_n^0)^2}}, \tag{14}$$

   where $p_n^0 = \pi n/L$ corresponds to the momentum of the free waves.

10. Show that in the continuous limit the expression can be rewritten to

$$M^{\text{quant}} \approx M + \frac{\sqrt{3}m}{4} + \frac{m}{\pi} \int_0^\infty \mathrm{d}y \left(\frac{1}{1+y^2} + \frac{2}{1+4y^2} \right) \sqrt{1+y^2}; \tag{15}$$

with $y = p/m$.

11. The integral is logarithmically divergent. Therefore, we can introduce a cut-off at $p = M_{UV}$. Show that the quantum-corrected domain wall mass can be approximated by

$$M^{\text{quant}} \approx M + \frac{3m}{2\pi} \ln \left(\frac{M_{UV}}{m} \right). \tag{16}$$