04 Problem Sheet

Freitag, 27. Oktober 2023 08:58

Problem 1:

(1)

- Newtonian limit: we analyse static configuration → time independent

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| \ll 1$$

additionally $\vec{v} \ll 1 \rightarrow \frac{d\vec{x}}{d\tau} = \frac{dt}{d\tau} \underbrace{\frac{d\vec{x}}{dt}}_{=\vec{v}} \ll \frac{dt}{d\tau}$

- geodesic equation:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

In Newtonian limit:

$$\frac{d^2 x^i}{d\tau^2} + \Gamma^i_{00} \left(\frac{dt}{d\tau}\right)^2 = 0$$

2x chain rule

$$\frac{d^2 x^i}{dt^2} = -\Gamma^i_{00}$$

- Christoffel symbols:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\beta g_{\nu\alpha} + \partial_\alpha g_{\nu\beta} - \partial_\nu g_{\alpha\beta} \right)$$

$$\Gamma^\mu_{00} \approx \frac{1}{2} \eta^{\mu\nu} \left(0 + 0 - \partial_\nu h_{00} \right) = -\frac{1}{2} \eta^{\mu\nu} \partial_\nu h_{00}$$

$$\Rightarrow \quad \frac{d^2 x^i}{dt^2} = -\frac{1}{2} \partial_i h_{00} \qquad (1)$$

- $$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \overbrace{\Gamma^{\rho}{}_{\mu\lambda} \Gamma^{\lambda}{}_{\nu\sigma}}^{\sim \mathcal{O}(h^2)} - \overbrace{\Gamma^{\rho}{}_{\nu\lambda} \Gamma^{\lambda}{}_{\mu\sigma}}^{\sim \mathcal{O}(h^2)}$$

$$R^{i}{}_{0j0} \approx \partial_j \Gamma^{i}{}_{00} + 0$$

$$R_{00} = R^{i}{}_{0i0} \approx - \partial_i \left(\frac{1}{2} \eta^{i\nu} \partial_\nu h_{00} \right) = \frac{1}{2} \vec{\nabla}^2 h_{00} \qquad (2)$$

- Einstein equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \overset{\sim\mathcal{O}(h)}{R} \, \underset{\eta_{\mu\nu}}{\cancel{g_{\mu\nu}}} = 8\pi G T_{\mu\nu}$$

$$R^{\mu}{}_{\mu} - \frac{1}{2} R \eta^{\mu}{}_{\mu} = 8\pi G T^{\mu}{}_{\mu}$$

$$R = -8\pi G T$$

$$\rightarrow R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T \eta_{\mu\nu} \right)$$

- With (1) and (2) we obtain for $\mu\nu = 00$:

$$\vec{\nabla}^2 V = 8\pi G \left(T_{00} - \frac{1}{2} T \right)$$

(2)

- Hilbert energy-momentum tensor:

$$T_{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - g_{\mu\nu} \mathcal{L}$$

- $$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{\lambda}{4} (\phi^2 - v^2)^2$$
$$= \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\lambda}{4} (\phi^2 - v^2)^2$$

$$= \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{\lambda}{4} (\Phi^2 - v^2)^2$$

$$\rightarrow T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \Phi \partial^\alpha \Phi + g_{\mu\nu} \frac{\lambda}{4} (\Phi^2 - v^2)^2$$

- $\Phi_{DW} = v \cdot \tanh\left(\frac{1}{2} m_h x\right)$ with $m_h = 2\sqrt{2\lambda}\, v$

$$T_{00} = \frac{1}{2} (\partial_x \Phi_{DW})^2 + \lambda (\Phi_{DW}^2 - v^2)^2 = \varepsilon_{DW}$$

$$T_{11} = \frac{1}{2} (\partial_x \Phi_{DW})^2 - \lambda (\Phi_{DW}^2 - v^2)^2 \underset{\uparrow \text{ by e.o.m.}}{=} 0$$

$$T_{22} = -\frac{1}{2} (\partial_x \Phi_{DW})^2 - \lambda (\Phi_{DW}^2 - v^2)^2 = -\varepsilon_{DW} = T_{33}$$

$$\rightarrow T_{\mu\nu} = \varepsilon_{DW} \, \mathrm{diag}(1, 0, -1, -1)$$

(3)

- $T = g^{\mu\nu} T_{\mu\nu} = T_{00} - T_{ii}$
  $= 3 \varepsilon_{DW}$

- $\vec{\nabla}^2 V = 8\pi G \left(\varepsilon_{DW} - \frac{3}{2} \varepsilon_{DW}\right)$
  $= -4\pi G \, \varepsilon_{DW}$

→ Because of the minus sign the gravitational field of the domain wall is repulsive

(4)

- $\varepsilon_{DW} = \frac{1}{2} (\partial_x \Phi_{DW})^2 + \lambda (\Phi_{DW}^2 - v^2)^2$

  $\overset{\text{e.o.m.}}{=} 2\lambda (\Phi_{DW}^2 - v^2)^2$

• $V(\vec{r}) = \int \left(- \frac{1}{4\pi|\vec{r}-\vec{r}'|}\right) \cdot \left(-8\pi G \lambda \left(\Phi_{DW}^2 - v^2\right)^2\right) \, d^3r'$

without loss of generality we can write $V(x,y,z) = V(x,0,0)$

$$= 2G\lambda v^4 \cdot 2\pi \int dx' \int_0^\infty d\rho' \, \frac{\rho'}{\sqrt{\rho'^2 + (x-x')^2}} \left(\tanh\left(\tfrac{1}{2} m_h x'\right)^2 - 1\right)^2$$

shift the potential

$$= \frac{\pi}{2} G m_h^2 v^2 \int dx' \left(\tanh\left(\tfrac{1}{2} m_h x'\right)^2 - 1\right)^2 \left(\cancel{\infty} + |x'-x|\right)$$

$$= \frac{\pi}{2} G v^2 \int d(m_h x') \left(\tanh\left(\tfrac{1}{2} m_h x'\right)^2 - 1\right)^2 |m_h x' - m_h x|$$

$$= \frac{\pi}{2} G v^2 \Bigg[\int_{-\infty}^{m_h x} d(m_h x') \left(\tanh\left(\tfrac{1}{2} m_h x'\right)^2 - 1\right)^2 (m_h x - m_h x') + \int_{m_h x}^{\infty} d(m_h x') \left(\tanh\left(\tfrac{1}{2} m_h x'\right)^2 - 1\right)^2 (m_h x' - m_h x) \Bigg]$$

Mathematica

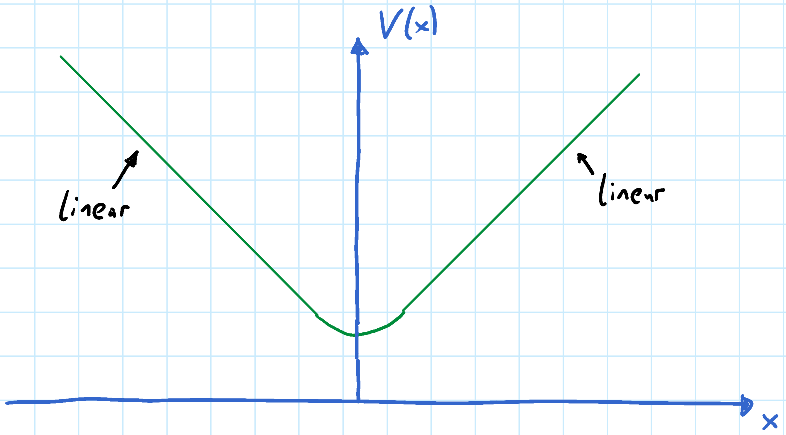
$$= \frac{2\pi}{3} G v^2 \left(\ln(16) + 4 \ln\left(\cosh\left(\frac{m_h x}{2}\right)\right) - \frac{2}{1+\cosh(m_h x)} \right)$$

• $\ln\left(\cosh\left(\frac{x}{2}\right)\right) = \ln\left(\frac{1}{2}\left(e^{-m_h x/2} + e^{m_h x/2}\right)\right)$

$$\xrightarrow{|x|\to\infty} \ln\left(\tfrac{1}{2}\right) + \frac{m_h |x|}{2}$$

$$\frac{2}{1+\cosh(m_h x)} \xrightarrow{|x|\to\infty} 0$$

$$\Rightarrow V(x) \xrightarrow{|x|\to\infty} \frac{\pi}{3} G v^2 \, m_h |x| + \frac{2\pi}{3} G v^2 \ln 8$$



Problem 2:

- $\mathcal{L} = \frac{1}{2} \partial_\mu \Phi^a \partial^\mu \Phi^a - V(\Phi)$, $d+1$ dimensions ($a = 1, \dots, n$)

energy $E = T + U$ with $T = \int d^d x \; \frac{1}{2} \partial_i \Phi^a \partial_i \Phi^a$

$U = \int d^d x \; V(\Phi)$

is minimized by a soliton solution.

(1)

- $\Phi_\lambda^a(x) = \Phi^a(\lambda x)$

$$T \longmapsto \int d^d x \; \frac{1}{2} \frac{\partial}{\partial x^i} \Phi^a(\lambda x) \frac{\partial}{\partial x^i} \Phi^a(\lambda x)$$

$$= \lambda^{2-d} \int d^d(\lambda x) \; \frac{1}{2} \frac{\partial}{\partial(\lambda x^i)} \Phi^a(\lambda x) \frac{\partial}{\partial(\lambda x^i)} \Phi^a(\lambda x)$$

$$= \lambda^{2-d} \; T$$

$$U \longmapsto \int d^d x \; V(\Phi(\lambda x)) = \lambda^{-d} \int d^d(\lambda x) \; V(\Phi(\lambda x)) = \lambda^{-d} U$$

$$\rightarrow E(\lambda) = \lambda^{2-d} T + \lambda^{-d} U$$

(2)

- The soliton extremises the energy and $\Phi_{cl}^a(\lambda x)$ is a soliton solution for $\lambda = 1$.

(3)

- $$\frac{dE}{d\lambda} = (2-d) \lambda^{1-d} \; T - d \; \lambda^{-d-1} \; U$$

$$\left. \frac{dE}{d\lambda} \right|_{\lambda=1} \stackrel{!}{=} 0$$

$$\rightarrow (2-d) \; T - d \; U = 0$$

- First, notice that $T, U \geq 0$

For $d > 2$: $\quad -|2-d| T = dU$

→ only possible solution is $T = U = 0$

→ trivial vacuum

- for $d = 2$: $\quad U = 0$

→ trivial vacuum

- for $d = 1$: $\quad T = U$

→ allowed with soliton solution

(4)

- $$\mathcal{L} = -\frac{1}{2} \operatorname{Tr}(G_{\mu\nu} G^{\mu\nu}) + \operatorname{Tr}\left((D_\mu \Phi)^\dagger (D^\mu \Phi)\right) - V(\Phi)$$

scale transformation: $\quad \Phi_\lambda(x) = \Phi_{cl}(\lambda x)$

$$A^i_\lambda(x) = \lambda A^i_{cl}(\lambda x)$$

- potential scales as before:

$$U \longmapsto \lambda^{-d} U$$

- $$T = \int d^d x \operatorname{Tr}\left((D_i \Phi(x))^\dagger D_i \Phi(x)\right)$$

$$\longmapsto \int d^d x \operatorname{Tr}\Big(\big(\partial_i \Phi^\dagger(\lambda x) - g\lambda [A_i^\dagger(\lambda x), \Phi^\dagger(\lambda x)]\big) \cdot \big(\partial_i \Phi(\lambda x) + g\lambda [A_i(\lambda x), \Phi(\lambda x)]\big)\Big)$$

$$= \lambda^{2-d} \int d^d(\lambda x) \operatorname{Tr}\Big(\Big(\frac{\partial}{\partial(\lambda x^i)} \Phi^\dagger(\lambda x) - g [A_i^\dagger(\lambda x), \Phi^\dagger(\lambda x)]\Big) \cdot \Big(\frac{\partial}{\partial(\lambda x^i)} \Phi(\lambda x) + g [A_i(\lambda x), \Phi(\lambda x)]\Big)\Big)$$

$= \lambda^{2-d} T$

- $G_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$

$\longmapsto \partial_i \lambda A_j(\lambda x) - \partial_j \lambda A_i(\lambda x) + \lambda^2 [A_i(\lambda x), A_j(\lambda x)]$

$= \lambda^2 \left[\frac{\partial}{\partial(\lambda x^i)} A_j(\lambda x) - \frac{\partial}{\partial(\lambda x^j)} A_i(\lambda x) + [A_i(\lambda x), A_j(\lambda x)] \right]$

$\rightarrow G = \int d^d x \, \frac{1}{2} \operatorname{Tr}(G_{ij} G_{ij})$

$\longmapsto \lambda^{4-d} G$

- $E(\lambda) = \lambda^{4-d} G + \lambda^{2-d} T + \lambda^{-d} U$

(5)

- $\left.\frac{dE}{d\lambda}\right|_{\lambda=1} = (4-d) G + (2-d) T - d U \overset{!}{=} 0$

Note that $G, T, U \geq 0$

- $d \geq 4$: $+ |4-d| G + |2-d| T + d U = 0$

Only $G = T = U = 0$ is allowed

- $d < 4$: $|4-d| G + |2-d| T = d U$

Also non-trivial solutions are allowed

For $d = 1, 2, 3$ we can have solitons.

Problem 3:

(1)

- $\langle |\Phi| \rangle = v \rightarrow \Phi = (\varphi + v) e^{iS}$

- $\mathcal{L} \supset \lambda ((\varphi + v)^2 - v^2)^2$

- $\mathcal{L} \supset \lambda\left((\varphi+v)^2 - v^2\right)^2$

$$= \lambda\left(\varphi^2 + 2v\varphi\right)^2 \supset 4\lambda v^2 \varphi^2 = \frac{1}{2}\left(8\lambda v^2\right)\varphi^2$$

$\rightarrow m_h = 2\sqrt{2\lambda}\, v$

- $D_\mu \langle|\Phi|\rangle = i g A_\mu v$

$\rightarrow \mathcal{L} \supset \left(D_\mu \langle|\Phi|\rangle\right)^* \left(D^\mu \langle|\Phi|\rangle\right) = g^2 v^2 A_\mu A^\mu$

$\rightarrow m_v = \sqrt{2}\, g v$

(2)

- $E = H$

for static case

$$\stackrel{\text{for static case}}{=} -\int d^2x\, \mathcal{L}$$

$$= \int d^2x \left(\frac{1}{4} F_{ij} F_{ij} + (D_i\Phi)^* (D^i \Phi) + \lambda \left(|\Phi|^2 - v^2\right)^2 \right)$$

- conditions for finite energy:

$$D_i \Phi \xrightarrow{r \to \infty} 0$$

$$|\Phi| \xrightarrow{r \to \infty} v$$

(3)

- $\Phi \xrightarrow{r \to \infty} v \cdot e^{in\Theta}$

$$D_i \Phi \xrightarrow{r \to \infty} v e^{in\Theta}\, in\, \partial_i \Theta + i g v A_i\, e^{in\Theta} \stackrel{!}{=} 0$$

$\rightarrow A_i \xrightarrow{r \to \infty} \frac{1}{g} n\, \partial_i \Theta$

- $\Theta = \arctan \frac{y}{x}$

$$\partial_x \Theta = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{y}{r}$$

$$\partial_x \Theta = \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{y}{r}$$

$$\partial_y \Theta = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{x}{r}$$

$$\rightarrow A_i \xrightarrow{r\to\infty} -\frac{1}{g}\, n\, \varepsilon_{ij} \frac{r^j}{r}$$

(4)

- magnetic flux:

$$\Phi = \int_S d\vec{S}\cdot\vec{B} = \oint_{\partial S} d\vec{l}\cdot\vec{A}$$

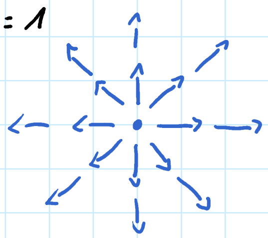
$$= \frac{n}{g} \oint_{\partial S} (\partial_i \Theta)\, dx^i$$

$$= \frac{n}{g} \int_0^{2\pi} d\Theta = \frac{2\pi n}{g}$$

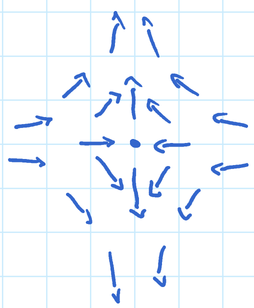
(5)

- $\begin{pmatrix} \mathrm{Re}\,\hat{\phi} \\ \mathrm{Im}\,\hat{\phi} \end{pmatrix} = \begin{pmatrix} \cos(n\Theta) \\ \sin(n\Theta) \end{pmatrix}$

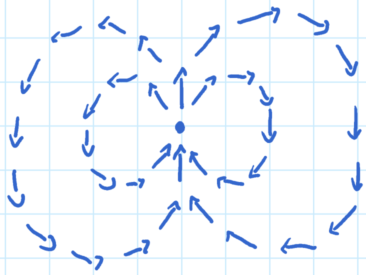
$n=1$



$n=-1$



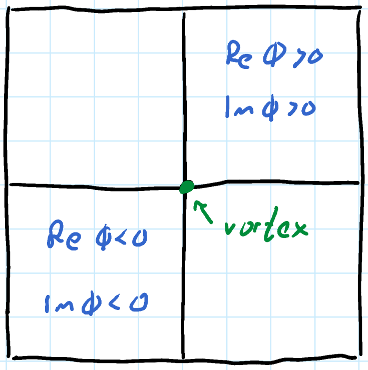
$n=2$



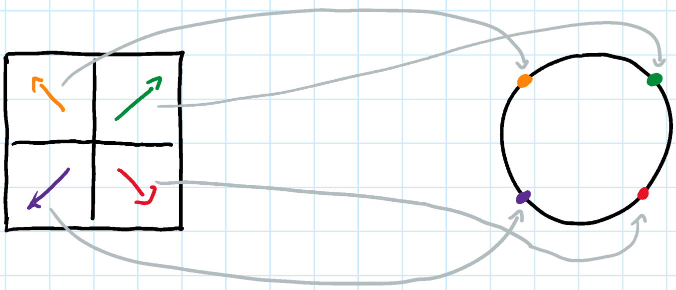
(6)

- We can have many causally disconnected areas that obtain a different vacuum expectation values. Note that in the given theory we have infinitely many vacuum expectation values.

- Imagine that one area obtains a value with $\mathrm{Re}\,\Phi, \mathrm{Im}\,\Phi > 0$ and another area obtains a value with $\mathrm{Re}\,\Phi, \mathrm{Im}\,\Phi < 0$, then there is always a point where the field Φ has to vanish. There, a vortex will appear:



- Around the vortex each point can be mapped to a point on the vacuum manifold:



Comment: A stable domain wall cannot exist in

Comment: A stable domain wall cannot exist in this theory. We will explain the reason on a later sheet.

(7)

- Yes, there can be a string solution.

"Instead of one vortex we have many vortices layered on top of each other."