03 Problem Sheet

Dienstag, 31. Oktober 2023 10:12

Problem 1:

Φ^6-theory: $\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \lambda \Phi^2(\Phi^2 - v^2)^2$

- Higgs mass: $\mathcal{L} \supset \lambda\,(h+v)^2 \cdot ((h+v)^2 - v^2)^2$

$$= \lambda\,(h^2 + 2vh + v^2)\cdot(h^2 + 2vh)^2$$

$$\supset 4\lambda v^4 h^2 \quad \rightarrow \quad m_h = 2\sqrt{2\lambda}\,v^2$$

(1)

- Bogomolny equation $\partial_x \Phi = \pm\sqrt{2V}$ (sheet 2)

$$\pm x = \int \frac{1}{\sqrt{2V}}\,d\Phi$$

$$= \int \frac{1}{\sqrt{2\lambda}}\,\frac{1}{\Phi(\Phi^2 - v^2)}\,d\Phi$$

$$= \int \frac{1}{\sqrt{2\lambda}}\,\frac{1}{\Phi^3\left(1 - \frac{v^2}{\Phi^2}\right)}\,d\Phi \qquad \text{with substitution: } \tilde{\Phi} = \pm\left(1 - \frac{v^2}{\Phi^2}\right)$$

$$\rightarrow d\tilde{\Phi} = \pm\frac{2v^2}{\Phi^3}\,d\Phi$$

$$= \int \frac{1}{m_h}\,\frac{1}{\tilde{\Phi}}\,d\tilde{\Phi}$$

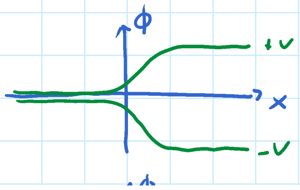
$$= \frac{1}{m_h}\ln\tilde{\Phi} \pm a$$

$$\rightarrow \tilde{\Phi} = e^{\pm m_h(x-a)}$$

- $+\left(1 - \frac{v^2}{\Phi^2}\right) = e^{\pm m_h(x-a)} \quad \rightarrow \quad \Phi = \frac{\pm v}{\sqrt{1 - e^{\pm m_h(x-a)}}}$ ↯ for $x = 0$

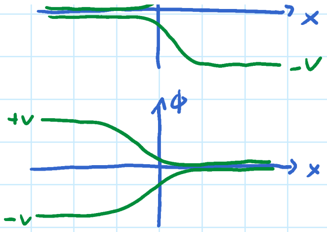
- $-\left(1 - \frac{v^2}{\Phi^2}\right) = e^{\pm m_h(x-a)}$

$$\rightarrow \Phi_{(0,\pm v)} = \frac{\pm v}{\sqrt{1 + e^{-m_h(x-a)}}}$$



$$\rightarrow \quad \Phi_{(0,\pm v)} = \frac{\pm v}{\sqrt{1+e^{-m_h(x-a)}}}$$

$$\Phi_{(\pm v,0)} = \frac{\pm v}{\sqrt{1+e^{+m_h(x-a)}}}$$



(2)

- We use the same trick as on sheet 2:

$$E = \int_{-\infty}^{\infty} \left(\frac{1}{2}(\partial_x \Phi)^2 + V \right) dx$$

$$= \int_{-\infty}^{\infty} 2V \, dx$$

$$= \int_0^v - \sqrt{2V} \, d\Phi$$

$$= \int_0^v - \sqrt{2\lambda} \, \Phi \, (\Phi^2 - v^2) \, d\Phi$$

$$= -\sqrt{2\lambda} \left(\frac{1}{4} v^4 - \frac{1}{2} v^4 \right) = \frac{\sqrt{2\lambda}}{4} v^4 = \frac{m_h v^2}{8}$$

Problem 2:

(1)

(a) $\Phi_{(0,v,0)} = \frac{1}{v} \Phi_{(0,v)}(x) \cdot \Phi_{(v,0)}(x-h)$

(b) $\Phi_{(v,0,v)} = \Phi_{(v,0)}(x) + \Phi_{(0,v)}(x-h)$

(c) $\Phi_{(v,0,-v)} = \Phi_{(v,0)}(x) + \Phi_{(0,-v)}(x-h)$

(2)

- We can show that the given bad choice

$$\Phi_{(v,0,v)}(x) = \frac{1}{v} \Phi_{(0,v)}(x) \, \Phi_{(-v,0)}(x-h) + v$$

does not satisfy the static field equation (Bogomolny equation) in the limit $h \to \infty$.

- Bogomolny eq. $\quad \partial_x \Phi_{DW} = \pm \sqrt{2V}$

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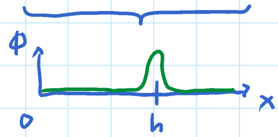
- $\partial_x\left(\frac{1}{v}\,\Phi_{(0,v)}(x)\,\Phi_{(-v,0)}(x-h)+v\right)$

$$= \frac{1}{v}\,\Phi'_{(0,v)}(x)\,\Phi_{(-v,0)}(x-h) + \frac{1}{v}\,\Phi_{(0,v)}(x)\,\underbrace{\Phi'_{(-v,0)}(x-h)}$$



$$\xrightarrow[x\ll h]{h\to\infty} -\Phi'_{(0,v)}(x)$$

- $\sqrt{2V} = \sqrt{2\lambda}\left(\frac{1}{v}\,\Phi_{(0,v)}(x)\,\Phi_{(-v,0)}(x-h)+v\right)$

$$\cdot\left(\frac{1}{v^2}\,\Phi^2_{(0,v)}(x)\,\Phi^2_{(-v,0)}(x-h)+2\,\Phi_{(0,v)}(x)\,\Phi_{(-v,0)}(x-h)\right)$$

$$\xrightarrow[x\ll h]{h\to\infty} \sqrt{2\lambda}\left(-\Phi_{(0,v)}(x)+v\right)\left(\Phi^2_{(0,v)}(x)-2v\,\Phi_{(0,v)}(x)\right)$$

$$= -\sqrt{2\lambda}\,\Phi_{(0,v)}(x)\left(\Phi^2_{(0,v)}(x)-v^2\right)$$

$$+\sqrt{2\lambda}\,v\left(\Phi^2_{(0,v)}(x)-2v\,\Phi_{(0,v)}(x)\right)-\sqrt{2\lambda}\,\Phi_{(0,v)}(x)\left(-2v\,\Phi_{(0,v)}(x)+v^2\right)$$

$$= -\sqrt{2V[\Phi_{(0,v)}(x)]}$$

$$+\sqrt{2\lambda}\,v\,\underbrace{\left(3\,\Phi^2_{(0,v)}(x)-3v\,\Phi_{(0,v)}(x)\right)}_{\neq 0}$$

→ $\Phi'_{(0,v)}(x) = \sqrt{2V[\Phi_{(0,v)}(x)]} - \sqrt{2}\,\lambda v\left(3\Phi^2_{(0,v)}(x)-3v\,\Phi_{(0,v)}(x)\right)$

→ Bogomolny eq. not satisfied and thus the given configuration is not static in the $h\to\infty$ limit

Comment 1: You can also calculate the energy of this configuration and you will of this configuration and you will notice that it is bigger than two times the energy of a single DW.

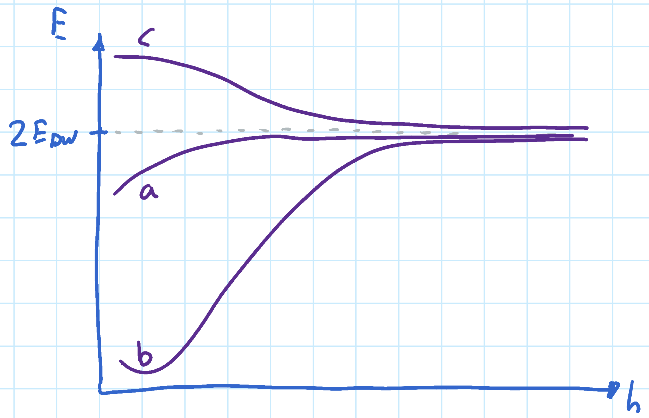
Comment 2: Another non-static solution configuration is

$$\Phi_{(v,0,v)} = \Phi_{(0,-v)}(x) + \Phi_{(0,v)}(x-h) + v.$$

In the tutorial I showed you what happens in the time evolution.

(3)

- For this problem, we will use Mathematica to avoid tedious calculations. The Mathematica notebook you can download on the webpage.



We observe that for (a) the energy is bigger for smaller distances → the DWs will repel. For (b) and (c) the energy is smaller for small distances → the DWs will attract.

(4)

- $\Phi_{(v,0,v)} = \Phi_{(v,0)}(x) + \Phi_{(0,v)}(x-h)$

(4)

- $\Phi_{(v,0,v)} = \Phi_{(v,0)}(x) + \Phi_{(0,v)}(x-h)$

- $\mathcal{L} \supset \frac{1}{2}\left[\partial_x\left(\Phi_{(v,0)}(x) + \Phi_{(0,v)}(x-h)\right)\right]^2$

$$= \underbrace{\frac{1}{2}\Phi'^2_{(v,0)}(x) + \frac{1}{2}\Phi'^2_{(0,v)}(x-h)}_{\text{gradient terms of single domain walls}} + \underbrace{\Phi'_{(v,0)}(x)\,\Phi'_{(0,v)}(x-h)}_{\text{interaction term}}$$

$$\supset \Phi'_{(v,0)}(x)\,\Phi'_{(0,v)}(x-h)$$

$$= \left(\frac{v\cdot m_h}{2}\left(1+e^{+m_h x}\right)^{-3/2}\cdot e^{m_h x}\right)\cdot\left(\frac{v\cdot m_h}{2}\left(1+e^{-m_h(x-h)}\right)^{-3/2}\cdot e^{-m_h(x-h)}\right)$$

$$= \frac{v^2 m_h^2}{4}\left(1+e^{m_h x}\right)^{-3/2}\left(1+e^{-m_h x}\cdot e^{m_h h}\right)^{-3/2} e^{m_h h}$$

$$\xrightarrow[x\ll h]{h\to\infty} \frac{v^2 m_h^2}{4}\left(1+e^{m_h x}\right)^{-3/2} e^{+\frac{3}{2}m_h x}\, e^{-\frac{3}{2}m_h h}\, e^{m_h h}$$

$$= (\ldots)\; e^{-\frac{1}{2}m_h h}$$

- $\mathcal{L} \supset -\lambda\left(\Phi_{(v,0)}(x) + \Phi_{(0,v)}(x-h)\right)^2\left(\left(\Phi_{(v,0)}(x)+\Phi_{(0,v)}(x-h)\right)^2 - v^2\right)^2$

$$= -\lambda\left(\Phi^2_{(v,0)}(x) + 2\Phi_{(v,0)}(x)\,\Phi_{(0,v)}(x-h) + \Phi^2_{(0,v)}(x-h)\right)$$
$$\cdot\left(\Phi^2_{(v,0)}(x) + 2\Phi_{(v,0)}(x)\,\Phi_{(0,v)}(x-h) + \Phi^2_{(0,v)}(x-h) - v^2\right)^2$$

$$\Phi_{(0,v)}(x-h) = \frac{v}{\sqrt{1+e^{-m_h(x-h)}}} \xrightarrow[x\ll h]{h\to\infty} v\cdot e^{\frac{1}{2}m_h x}\, e^{-\frac{1}{2}m_h h}$$

We want to find the term that has the smallest suppression. Here, these are terms where $\Phi_{(0,v)}(x-h)$ enters only once.

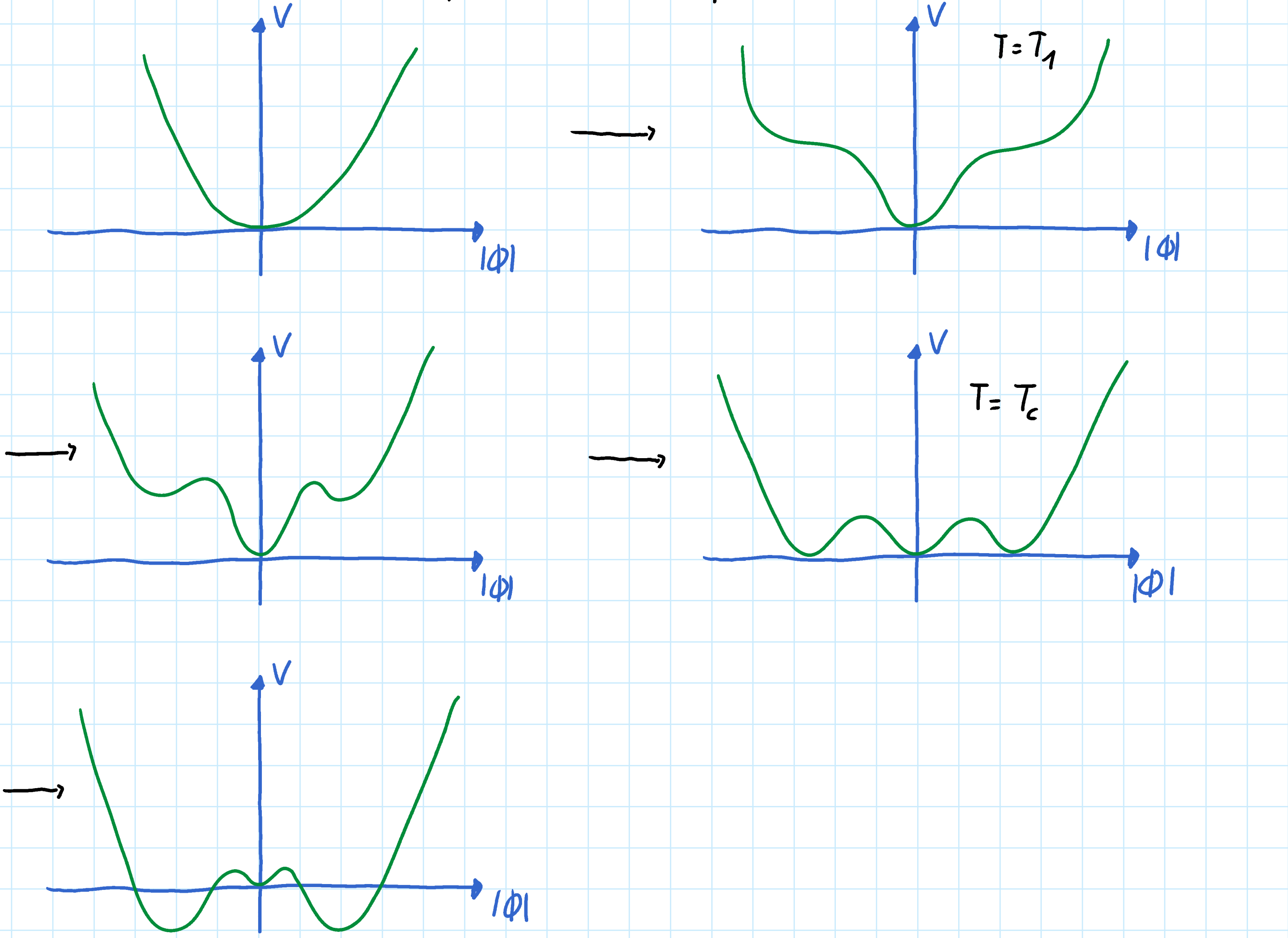
$\Rightarrow$ Summarized we can say that the interaction is suppressed with $e^{-\frac{1}{2} m_h h}$

Problem 3:

- $V(\Phi) = m^2(T)\,|\Phi|^2 + \dfrac{m_0^2}{v\sigma^2}\,|\Phi|^4 \ln \dfrac{|\Phi|^2}{\sigma^2}$

  with $m^2(T) = m_0^2 + \frac{1}{4} g^2 T^2$

- With decreasing temperature the potential behaves as follows



- Rewrite potential:

$$V(\Phi) = m^2(T)\,|\Phi|^2 + \frac{m_0^2}{v\sigma^2}\,|\Phi|^4 \ln \frac{|\Phi|^2}{\sigma^2}$$

$$= m_0^2 \sigma^2 \left(1 + \frac{g^2T^2}{4m_0^2}\right)\left[\frac{|\Phi|^2}{\sigma^2} + \frac{1}{v\cdot\left(1 + g^2T^2\right)}\,\frac{|\Phi|^4}{\sigma^4} \ln \frac{|\Phi|^2}{\sigma^2}\right]$$

$$= m_0^2 \sigma^2 \left(1 + \frac{g^2 T^2}{4 m_0^2}\right) \underbrace{\left[\frac{|\Phi|^2}{\sigma^2} + \frac{1}{v\left(1 + \frac{g^2 T^2}{4 m_0^2}\right)} \frac{|\Phi|^4}{\sigma^4} \ln \frac{|\Phi|^2}{\sigma^2} \right]}_{\text{overall shape of the potential depends only on this part}}$$

→ We will analyse the function:

$$f(x) = x^2 + \frac{1}{\tilde{v}} x^4 \ln x^2 \qquad \text{where} \quad \tilde{v} = v \cdot \left(1 + \frac{g^2 T^2}{4 m_0}\right)$$

- $T = T_1$:

$$f'(x) = 2x + \frac{4}{\tilde{v}} x^3 \ln x^2 + \frac{2}{\tilde{v}} x^3 = 0$$

$$\rightarrow \tilde{v} + 2 x^2 \ln x^2 + x^2 = 0 \qquad (1)$$

$$f''(x) = 2 + \frac{12}{\tilde{v}} x^2 \ln x^2 + \frac{8}{\tilde{v}} x^2 + \frac{6}{\tilde{v}} x^2 = 0$$

$$\rightarrow \tilde{v} + 6 x^2 \ln x^2 + 7 x^2 = 0 \qquad (2)$$

$(2) - 3 \cdot (1)$: $\quad -2\tilde{v} + 4x^2 = 0$

$$\rightarrow x = \pm \sqrt{\frac{\tilde{v}}{2}}$$

In (1): $\quad \tilde{v} + 2 \cdot \frac{\tilde{v}}{2} \cdot \ln\left(\frac{\tilde{v}}{2}\right) + \frac{\tilde{v}}{2} = 0$

$$\rightarrow \tilde{v}_1 = 2 \cdot e^{-3/2} = v \left(1 + \frac{g^2 T_1^2}{4 m_0^2}\right)$$

$$\rightarrow T_1 = \frac{2 m_0}{g} \sqrt{\frac{2}{v} \cdot e^{-3/2} - 1}$$

- $T = T_c$:

$$f'(x) = 0 \quad \rightarrow \quad (1)$$

$$f(x) = x^2 + \frac{1}{\tilde{v}} x^4 \ln x^2 = 0$$

$$\rightarrow \tilde{v} + x^2 \ln x^2 = 0 \qquad (3)$$

$2\cdot(3) - (1)$: $\quad \tilde{v} - x^2 = 0 \quad \rightarrow \quad x = \pm\sqrt{\tilde{v}}$

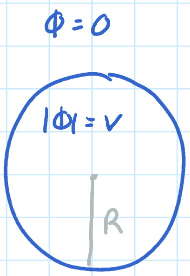
In (1): $\quad \tilde{v} + \tilde{v} \ln \tilde{v} = 0$

$$\tilde{v}_c = \frac{1}{e} = v\left(1 + \frac{g^2 T_c^2}{4 m_0^2}\right)$$

$$\rightarrow T_c = \frac{2 m_0}{g} \sqrt{\frac{1}{e \cdot v} - 1}$$

- At $T = T_c$ the overall shape of the Coleman-Weinberg potential is very similar to the sextic potential of problem 1. For both potentials, the symmetric and Higgs phases can co-exist.

Problem 4:



$$\Phi_{bubble} = \Phi_{(\pm v, 0)}(r - R)$$

(1)

- energy of domain wall: $E = \gamma \cdot E_{DW}$ $\quad$ DW tension

$$= \gamma \cdot 2\pi R(t) \cdot \sigma_{DW}$$

$$= E_0 = 2\pi R_0 \cdot \sigma_{DW}$$

$$\rightarrow R_0 = \gamma \cdot R(t) = \frac{1}{\sqrt{1 - \dot{R}(t)^2}} R(t)$$

$$\rightarrow \quad R_0 = \gamma \cdot R(t) = \frac{1}{\sqrt{1-\dot{R}(t)^2}} R(t)$$

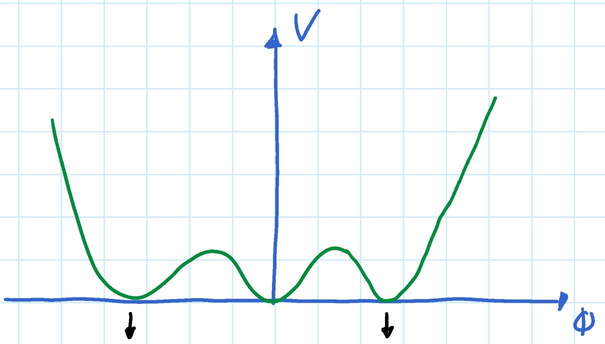
$$R_0^2 \left(1 - \dot{R}(t)^2\right) = R(t)^2$$

$$1 = \left(\frac{R(t)}{R_0}\right)^2 + \dot{R}(t)^2$$

solution can be guessed:

$$R(t) = R_0 \cos\left(\frac{t}{R_0}\right) \qquad \text{for} \quad t \in [0, R_0]$$

(2)



- A growing bubble means that the Higgs phase has to be preferred. This can be achieved by shifting the minima $\Phi \neq 0$ down.

A way to realize this is to add a Φ^2 term:

$$V(\Phi) = \lambda (\Phi^2 - v^2)^2 \Phi^2 - \beta \Phi^2 \qquad \text{with} \quad \beta > 0$$

(3)

- $T > T_1$: Only the symmetric phase exists.

- $T_c < T < T_1$: Further minima appear. Through fluctuations small bubbles in the Higgsed phase can small bubbles in the Higgsed phase can emerge. However, the will collapse immediately, because the minimum $\Phi = 0$ is lower and additionally there is a tension.

- $T < T_c$: Minima $\Phi \neq 0$ are below the $\Phi = 0$ minimum. Now, the vacuum bubbles that are big enough will grow and small bubbles will still collapse.

- $T \ll T_c$: The whole space will be filled with the Higgs vacuum.

(4)

- Two expanding bubbles can collide. At the collision but also after the collision, gravitational wave are emitted. They may be detectable in the future. NanoGrav already gave results in this direction.