Problem 1:

- $$\mathcal{L} = -\frac{1}{4} \overset{A_\mu}{F_{\mu\nu}} F^{\mu\nu} - \frac{1}{4} \overset{B_\mu}{G_{\mu\nu}} G^{\mu\nu} + (D_\mu \Phi_1)^* (D^\mu \Phi_1) + (D_\mu \Phi_2)^* (D^\mu \Phi_2)$$
$$+ m^2 (|\Phi_1|^2 + |\Phi_2|^2) - \lambda (|\Phi_1|^2 + |\Phi_2|^2)^2$$

$$D_\mu \Phi_{1/2} = \partial_\mu \Phi_{1/2} - ia A_\mu \Phi_{1/2} - ib B_\mu \Phi_{1/2}$$

(1)

- $$|\Phi_1|^2 + |\Phi_2|^2 = Re(\Phi_1)^2 + Im(\Phi_1)^2 + Re(\Phi_2)^2 + Im(\Phi_2)^2$$
$$= \begin{pmatrix} Re\,\Phi_1 \\ Im\,\Phi_1 \\ Re\,\Phi_2 \\ Im\,\Phi_2 \end{pmatrix}^T \begin{pmatrix} Re\,\Phi_1 \\ Im\,\Phi_1 \\ Re\,\Phi_2 \\ Im\,\Phi_2 \end{pmatrix}$$

→ Transformation $\begin{pmatrix} Re\,\Phi_1 \\ Im\,\Phi_1 \\ Re\,\Phi_2 \\ Im\,\Phi_2 \end{pmatrix} \longmapsto O \begin{pmatrix} Re\,\Phi_1 \\ Im\,\Phi_1 \\ Re\,\Phi_2 \\ Im\,\Phi_2 \end{pmatrix}$ with $O^T O = \mathbb{1}$

keeps Lagrangian invariant.

→ global symmetry is $O(4)$

(2)

- $$D_\mu \Phi_{1/2} = \partial_\mu \Phi_{1/2} - ia A_\mu \Phi_{1/2} - ib B_\mu \Phi_{1/2}$$
$$= \partial_\mu \Phi_{1/2} - i Z_\mu \Phi_{1/2}$$

where $Z_\mu = a A_\mu + b B_\mu$

- The local symmetry of the theory is

$$\Phi_{1/2} \longmapsto e^{i\alpha(x)} \Phi_{1/2}$$
$$Z_\mu \longmapsto Z_\mu + \partial_\mu \alpha$$

→ The gauge symmetry is $U(1)$

→ The gauge and the global symmetries are are not always the same

- BUT: Here the definition of the gauge symmetry is not so exact. Since there is another field $\bar{Z}_\mu := aA_\mu - bB_\mu$ that is completely decoupled from $\Phi_{1/2}$, we can say that

$$\bar{Z}_\mu \longmapsto \bar{Z}_\mu + \partial_\mu \alpha$$

is another $U(1)$ symmetry and thus the gauge symmetry is

$$U(1) \times U(1)$$

⌈ Why $\bar{Z}_\mu \longmapsto \bar{Z}_\mu + \partial_\mu \alpha$ corresponds to $U(1)$?

For $SU(N)$ we have

$$\bar{Z}_\mu \longmapsto U^\dagger \bar{Z}_\mu U + i U \partial_\mu U^\dagger.$$

For $N=1$, this is just $\bar{Z}_\mu \longmapsto \bar{Z}_\mu + \partial_\mu \alpha$ ⌋

(3)

- $V = \lambda(|\Phi_1|^2 + |\Phi_2|^2)^2 - m^2(|\Phi_1|^2 + |\Phi_2|^2)$

$$\frac{\partial V}{\partial \Phi^*_{1/2}} = 2\lambda(|\Phi_1|^2 + |\Phi_2|^2)\Phi_{1/2} - m^2 \Phi_{1/2} \overset{!}{=} 0$$

From here we obtain 2 possibilities:

(I) $|\Phi_1|^2 + |\Phi_2|^2 = \frac{m^2}{2\lambda} \equiv v^2$

$$(\mathrm{I}) \quad |\Phi_1|^2 + |\Phi_2|^2 = \frac{m^2}{2\lambda} \equiv v^2$$

$$(\mathrm{II}) \quad \Phi_1 = 0 \quad , \quad \Phi_2 = 0$$

- Case (II) corresponds to a local maximum
  → not relevant for us

- Case (I) corresponds to the minima and therefore gives the vacuum manifold:

$$\mathcal{M} = \{ \Phi_1, \Phi_2 \in \mathbb{C} : |\Phi_1|^2 + |\Phi_2|^2 = v^2 \}$$

- $\mathrm{Re}(\Phi_1)^2 + \mathrm{Im}(\Phi_1)^2 + \mathrm{Re}(\Phi_2)^2 + \mathrm{Im}(\Phi_2)^2 = v^2$
  → the geometric form of the vacuum manifold is a 3-sphere

(4)

- $v = \frac{m}{\sqrt{2\lambda}}$ (see part (3))

(5)

(a)

- $\Phi_1 = v + \varphi_{1r} + i\varphi_{1i} \qquad \Phi_2 = \varphi_{2r} + i\varphi_{2i}$

$$\mathcal{L} \supset m^2 \left(|\Phi_1|^2 + |\Phi_2|^2\right) - \lambda \left(|\Phi_1|^2 + |\Phi_2|^2\right)^2$$

$$= m^2 \left(v^2 + 2v\varphi_{1r} + \varphi_{1r}^2 + \varphi_{1i}^2 + \varphi_{2r}^2 + \varphi_{2i}^2 \right)$$

keep only second order terms

$$- \lambda \left(v^2 + 2v\varphi_{1r} + \varphi_{1r}^2 + \varphi_{1i}^2 + \varphi_{2r}^2 + \varphi_{2i}^2 \right)^2$$

$$\supset m^2 \left(\varphi_{1r}^2 + \varphi_{1i}^2 + \varphi_{2r}^2 + \varphi_{2i}^2\right)$$

$$- \underbrace{4v^2\lambda}_{=2m^2} \varphi_{1r}^2 - \underbrace{2\lambda v^2}_{=m^2} \left(\varphi_{1r}^2 + \varphi_{1i}^2 + \varphi_{2r}^2 + \varphi_{2i}^2\right)$$

$$- \underbrace{4v^2\lambda}_{=2m^2} \varphi_{1r}^2 - \underbrace{2\lambda v^2}_{=m^2}(\varphi_{1r}^2 + \varphi_{1i}^2 + \varphi_{2r}^2 + \varphi_{2i}^2)$$

$$= -2m^2 \varphi_{1r}^2$$

$\rightarrow$ one massive particle φ_{1r} with mass $m_{1r} = 2m$ and three massless particles φ_{1i}, φ_{2r}, φ_{2i}.

(b)

- $\Phi_1 = \frac{iv}{\sqrt{2}} + \varphi_{1r} + i\varphi_{1i}$, $\Phi_2 = \frac{v}{\sqrt{2}} + \varphi_{2r} + i\varphi_{2i}$

$$\mathcal{L} \supset m^2 \left(\varphi_{1r}^2 + \varphi_{1i}^2 + \frac{v^2}{2} + \sqrt{2}v\varphi_{1i} + \varphi_{2r}^2 + \varphi_{2i}^2 + \frac{v^2}{2} + \sqrt{2}v\varphi_{2r} \right)$$
$$- \lambda \left(\varphi_{1r}^2 + \varphi_{1i}^2 + \varphi_{2r}^2 + \varphi_{2i}^2 + v^2 + \sqrt{2}v(\varphi_{1i} + \varphi_{2r}) \right)^2$$

keep only second order terms

$$\supset m^2 (\varphi_{1r}^2 + \varphi_{1i}^2 + \varphi_{2r}^2 + \varphi_{2i}^2)$$
$$- \lambda \left(2v^2 (\varphi_{1r}^2 + \varphi_{1i}^2 + \varphi_{2r}^2 + \varphi_{2i}^2) + 2v^2 (\varphi_{1i} + \varphi_{2r})^2 \right)$$

$$= -m^2 (\varphi_{1i} + \varphi_{2r})^2$$

$\rightarrow$ one massive particle $\frac{1}{\sqrt{2}}(\varphi_{1i} + \varphi_{2r})$ with mass $2m$ and three massless particles φ_{1r}, φ_{2i}, $\frac{1}{\sqrt{2}}(\varphi_{1i} - \varphi_{2r})$

$\rightarrow$ mass spectrum is the same

Note that with the normalization above we have

$$\varphi_{1r}^2 + \varphi_{2i}^2 + \left(\frac{1}{\sqrt{2}}(\varphi_{1i} + \varphi_{2r}) \right)^2 + \left(\frac{1}{\sqrt{2}}(\varphi_{1i} - \varphi_{2r}) \right)^2$$
$$= \varphi_{1r}^2 + \varphi_{2i}^2 + \varphi_{1i}^2 + \varphi_{2r}^2$$

(6)

(a)

- $\Phi_1 = v + \varphi_{1r} + i\varphi_{1i}$

$$\mathcal{L} \supset (\partial_\mu v + ia A_\mu v + ib B_\mu v)(\partial^\mu v - ia A^\mu v - ib B^\mu v)$$
$$= v^2 (a A_\mu + b B_\mu)^2$$

$\rightarrow$ one massive gauge boson $(a A_\mu + b B_\mu)$ with mass $\sqrt{2}\, v$ and one massless gauge boson $(a A_\mu - b B_\mu)$

(b)

- $\Phi_1 = \frac{iv}{\sqrt{2}} + \varphi_{1r} + i\varphi_{1i}$, $\Phi_2 = \frac{v}{\sqrt{2}} + \varphi_{2r} + i\varphi_{2i}$

$$\mathcal{L} \supset \frac{v^2}{2} (a A_\mu + b B_\mu)^2 + \frac{v^2}{2} (a A_\mu + b B_\mu)$$

$\rightarrow$ one massive gauge boson $(a A_\mu + b B_\mu)$ with mass $\sqrt{2}\, v$ and one massless gauge boson $(a A_\mu - b B_\mu)$

$\rightarrow$ the spectrum is the same

Comment 1: It makes sense that the massive gauge field is always the same, because only $a A_\mu + b B_\mu$ was coupled to the scalar field.

Comment 2: In (5) and (6), finding the massive and massless states was rather easy. In more complicated scenarios you can write down a mass matrix:

$$v^2(aA_\mu + bB_\mu)^2$$

$$= v^2a^2 A_\mu A^\mu + 2v^2ab\, A_\mu B^\mu + v^2b^2 B_\mu B^\mu$$

$$= \frac{1}{2}\begin{pmatrix} A_\mu & B_\mu \end{pmatrix} \begin{pmatrix} 2v^2a^2 & 2v^2ab \\ 2v^2ab & 2v^2b^2 \end{pmatrix} \begin{pmatrix} A^\mu \\ B^\mu \end{pmatrix}$$

diagonalizing this matrix gives the mass spectrum.

Problem 2:

(1)

- $\mathbb{Z}_2$ symmetry: $\phi \longmapsto -\phi$
- vacuum expectation value $\langle\phi\rangle = \pm v$

  $\phi = v + h$

  $\rightarrow \mathcal{L} \supset -\lambda\left((v+h)^2 - v^2\right)^2 \supset -4\lambda v^2 h^2$

  $\rightarrow$ the Higgs boson mass is $m_h = 2\sqrt{2\lambda}\, v$

(2)

- $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0$

  $\partial_\mu\partial^\mu\phi + \frac{\partial V}{\partial\phi} = 0$

- static field equations $\to \partial_t \Phi = 0$

$$\to \partial_x^2 \Phi = \frac{\partial V}{\partial \Phi}$$

- energy for static configuration is

$$E = \int dx \left(\frac{1}{2} (\partial_x \phi)^2 + V \right)$$

energy is finite if

$$\partial_x \Phi \xrightarrow{x \to \infty} 0$$

$$V \xrightarrow{x \to \infty} 0 \qquad (\text{i.e. } \Phi \xrightarrow{x \to \infty} v)$$

- Multiplying with $\frac{\partial \phi}{\partial x}$

$$\partial_x \Phi \; \partial_x^2 \Phi = \frac{\partial \Phi}{\partial x} \frac{\partial V}{\partial \Phi}$$

$$\frac{1}{2} \partial_x (\partial_x \Phi)^2 = \frac{\partial V}{\partial x}$$

$$(\partial_x \Phi)^2 = 2V + \text{const}$$

$\uparrow = 0$ since finite energy solution requires $\partial_x \Phi \xrightarrow{x \to \infty} 0$, $V \xrightarrow{x \to \infty} 0$

$$\to \partial_x \Phi = \pm \sqrt{2V}$$

(3)

- $V = 0$ for $\Phi = \pm v$

$\to$ trivial solutions are $\Phi = \pm v$

- non-trivial solutions satisfy

$$\Phi \xrightarrow{x \to -\infty} \mp v \quad , \quad \Phi \xrightarrow{x \to +\infty} \pm v$$

- $$\frac{d\phi}{dx} = \pm\sqrt{2V}$$

$$\rightarrow \pm x = \int \frac{1}{\sqrt{2V}}\, d\phi$$

$$= \int \frac{1}{\sqrt{2\lambda}} \quad \frac{1}{\phi^2 - v^2}\, d\phi$$

$$= -\frac{1}{v\sqrt{2\lambda}} \operatorname{arctanh}\left(\frac{\phi}{v}\right) \pm a$$

some constant

↑ sign is convention

$$\rightarrow \phi_{DW} = \pm v \tanh\left(v\sqrt{2\lambda}\,(x-a)\right)$$

$$= \pm v \tanh\left(\tfrac{1}{2} m_h (x-a)\right)$$

↑ position of DW

(4)

- $$E = \int_{-\infty}^{\infty} \left(\frac{1}{2}(\partial_x \phi)^2 + V \right) dx \qquad \text{with Bogomolny equation}$$

$$= \int_{-\infty}^{\infty} 2V\, dx$$

$$= \int_{\mp v}^{\pm v} 2V \frac{1}{\frac{d\phi}{dx}}\, d\phi \qquad \text{with Bogomolny equation}$$

$$= \int_{\mp v}^{\pm v} \mp\sqrt{2V}\, d\phi$$

$$= \int_{\mp v}^{\pm v} \mp\sqrt{2\lambda}\,(\phi^2 - v^2)\, d\phi$$

$$= \frac{4}{3}\sqrt{2\lambda}\, v^3 = \frac{2}{3} m_h v^2$$

(5)

- $$\partial_\mu J^\mu = \frac{1}{2v}\, \overbrace{\varepsilon^{\mu\nu}}^{\text{anti-sym.}}\, \underbrace{\partial_\mu \partial_\nu}_{\text{sym.}} \phi = 0$$

- $Q = \int dx\, J^0 = \frac{1}{2v} \int dx\, \varepsilon^{0\nu} \partial_\nu \phi$

$$= \frac{1}{2v} \int dx\, \partial_x \phi$$

$$= \frac{1}{2} \int dx\, \partial_x \tanh\left(\tfrac{1}{2} m_h x\right)$$

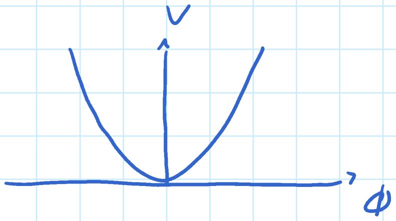
$$= \frac{1}{2} \left(\tanh(\infty) - \tanh(-\infty) \right) = 1$$

Problem 3:

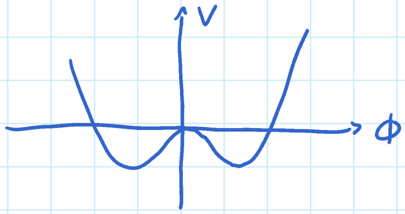
(1)

- $V = \lambda \phi^4 + \mu^2(T) \phi^2$

$T > T_c \rightarrow \mu^2(T) > 0$



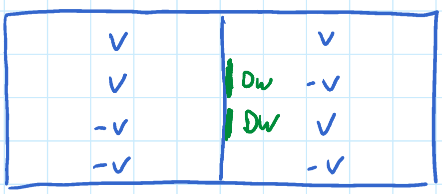
$T < T_c \rightarrow \mu^2(T) < 0$



$\rightarrow$ symmetry is broken for $T < T_c$

(2)

- There are two possible vacuum expectation values $\langle \phi \rangle = \pm v$.



$\rightarrow$ probability 50%

(3)

- there are $2^4 = 16$ possible configurations.

Only two states don't give a domain wall:

$\pm v$	$\pm v$
$\pm v$	$\pm v$

$\rightarrow$ probability that a domain wall forms is

$$\frac{14}{16} = 87{,}5\,\%$$