Topological Defects

Problem Sheet 2

30 October 2023

1. Symmetry Breaking with two Scalar Fields

Let us consider a gauge theory with two complex scalar fields ϕ_1 and ϕ_2. Furthermore, we have two gauge fields A_μ and B_μ. The Lagrangian is given by

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + (D_\mu\phi_1)^*(D^\mu\phi_1) + (D_\mu\phi_2)^*(D^\mu\phi_2) \\ &+ m^2\left(|\phi_1|^2 + |\phi_2|^2\right) - \lambda\left(|\phi_1|^2 + |\phi_2|^2\right)^2,\end{aligned} \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ and $D_\mu\phi_{1/2} = \partial_\mu\phi_{1/2} - iaA_\mu\phi_{1/2} - ibB_\mu\phi_{1/2}$.

1. What is the global symmetry of the theory ? Write down the corresponding transformations for ϕ_1 and ϕ_2.
2. What is the gauge symmetry of the theory ? Write down the local transformations that keep the Lagrangian invariant.
3. Find the vacuum manifold, i.e. the manifold of all values of the scalar fields that minimize the potential. Which geometric form has the vacuum manifold ?

Due to the symmetry of the theory, we can choose an arbitrary ground state. We will consider the following two cases :

$$\begin{aligned}&(a)\ \phi_1 = v, &&\phi_2 = 0,\\ &(b)\ \phi_1 = \frac{iv}{\sqrt{2}}, &&\phi_2 = \frac{v}{\sqrt{2}}.\end{aligned}$$

4. Find an expression for v.
5. If we consider the global case, i.e. we ignore the gauge fields, what is the mass spectrum of the scalar fields in both cases (a) and (b) ?
6. Find the mass spectrum of the gauge fields in both cases (a) and (b) by expanding the appropriate terms in the Lagrangian around the vacuum. How many massive gauge bosons do you obtain in each case ?

2. The Domain Wall

Consider a theory in $1+1$ dimensions with one real scalar field ϕ. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \lambda(\phi^2 - v^2)^2. \tag{2}$$

1. What is the symmetry of this theory and what is the Higgs boson mass ?

2. Find the static field equation of the theory and show that for finite energy solutions, it can be rewritten to the Bogomolny equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \pm\sqrt{2V(\phi)}. \tag{3}$$

3. Solve the Bogomolny equation to find the domain wall solutions. What are the trivial solutions to this equation ?
4. Calculate the total energy of a domain wall.
5. Show that the topological current

$$J^{\mu} = \frac{1}{2v}\varepsilon^{\mu\nu}\partial_{\nu}\phi \tag{4}$$

is conserved and calculate the topological charge for a domain wall.

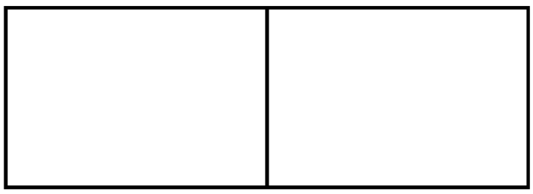
3. Second-Order Phase Transition and Kibble Mechanism

Instead of $1+1$ dimensions, we will consider a theory in $2+1$ dimensions. The Lagrangian has the form

$$\mathcal{L} = \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - \mu^2(T)\phi^2 - \lambda\phi^4. \tag{5}$$

Through thermal corrections, the ϕ^2 term can become temperature dependent. We captured this fact in the coupling constant $\mu^2(T)$. For $T > T_c$, where T_c is some critical temperature, $\mu^2(T) > 0$. For $T < T_c$, we have $\mu^2(T) < 0$.

1. For which temperatures the symmetry is broken ?
2. When the universe expands, the temperature cools down. Now take two regions that are causally disconnected when the temperature reaches its critical value. What is the probability that a domain wall forms ? For simplicity take the system of the form



3. What is the probability that a domain wall forms if you have four causally disconnected regions of the form