Sheet 8:

Hand-out: Tuesday, Dec. 12, 2023; Solutions: Tuesday, Dec. 19, 2023

Problem 1 Flux insertion – counts as two [a-d / e-g]

In this problem we consider non-interacting spin-less particles on a ring (or a torus) and couple them to a $U(1)$ gauge flux through the ring (or through one cycle of the torus).

(1.a) We start by considering a one-dimensional lattice on a ring of length L. The simplest way to introduce a total $U(1)$ gauge flux Φ is by the Hamiltonian:

$$\hat{\mathcal{H}}(\Phi) = -t \sum_{j=1}^{L-1} \left(\hat{c}_{j+1}^\dagger \hat{c}_j + \text{h.c.} \right) - t \left(e^{i\Phi} \hat{c}_1^\dagger \hat{c}_L + e^{-i\Phi} \hat{c}_L^\dagger \hat{c}_1 \right). \tag{1}$$

For which values of Φ is this Hamiltonian translationally invariant? How are eigenstates at $\Phi = 0$ and $\Phi = 2\pi$ related to one another?

(1.b) Find a unitary gauge transformation

$$\hat{U} = \exp \left[-i \sum_{j=1}^{L} \varphi_j \hat{n}_j \right] \tag{2}$$

such that

$$\tilde{\mathcal{H}}(\Phi) = \hat{U}^\dagger \hat{\mathcal{H}}(\Phi) \hat{U} \tag{3}$$

is translationally invariant (make an appropriate choice of φ_j and calculate $\tilde{\mathcal{H}}(\Phi)$ explicitly!). How are eigenstates at $\Phi = 0$ and $\Phi = 2\pi$ related?

(1.c) Using Fourier transformations, derive all eigenenergies $E_n(\Phi)$ of $\tilde{\mathcal{H}}(\Phi)$ for general values of Φ. Show that the corresponding eigenstates are plane waves with momentum

$$k_n = \frac{2\pi}{L} n, \qquad n = 1 ... L, \tag{4}$$

and show that eigenenergies are related as:

$$E_n(\Phi + 2\pi) = E_{n+1}(\Phi). \tag{5}$$

(1.d) Now consider an initial eigenstate $|\Psi_0(\Phi)\rangle$ of $\tilde{\mathcal{H}}(\Phi)$ for $\Phi = 0$ with N particles with momenta k_{n_m}, where $m = 1 ... N$ labels the particles and $n_m \in \{1, 2, ..., L\}$. Express the total momentum P_x of this state in terms of the momenta k_{n_m}.

(1.e) Next, assume that Φ is adiabatically increased from $\Phi = 0$ to $\Phi = 2\pi$, such that the quantum numbers k_{n_m} cannot change. Accordingly, P_x cannot change. Show that the new eigenstate $|\Psi_1\rangle = |\Psi_0(\Phi = 2\pi)\rangle$ of $\tilde{\mathcal{H}}(\Phi = 2\pi)$ is related to $|\Psi_0(\Phi = 0)\rangle$ by a gauge transformation $\hat{V}$:

$$|\Psi_1\rangle = \hat{V}^\dagger |\Psi_0(\Phi = 0)\rangle, \qquad \hat{V} = \exp\left[-i \sum_{j=1}^{L} \vartheta_j \hat{n}_j\right] \tag{6}$$

for appropriately chosen values of ϑ_j. *Hint:* Show that $\tilde{\mathcal{H}}(\Phi = 2\pi)$ and $\tilde{\mathcal{H}}(\Phi = 0)$ are related by the gauge transformation $\hat{V}$.

(1.f) Show that $|\Psi_1\rangle$ is also an eigenstate of $\tilde{\mathcal{H}}(\Phi = 0)$ but with momentum:

$$P'_x = P_x + \frac{2\pi}{L} N \mod 2\pi. \tag{7}$$

Hint: Use the relation from (1.d).

(1.g) Generalize your results from above for a higher-dimensional system on a $L_x \times L_y$ torus and show that

$$P'_x = P_x + \frac{2\pi}{L_x} N \mod 2\pi \tag{8}$$

when flux Φ_x is adiabatically introduced through the x-cycle of the torus. Here N still denotes the total particle number in the higher-dimensional system.

Problem 2 Effective field theory of fractional quantum Hall systems

In this problem, we explore topological aspects of fractional quantum systems using an effective field theory formalism. The field theory of a fractional quantum Hall system is described by the Chern-Simons Lagrangian

$$\mathcal{L} = \frac{1}{4\pi} \epsilon^{\mu\nu\lambda} \, \underline{a}_\mu^T \underline{\underline{K}} \partial_\nu \underline{a}_\lambda \; - \frac{e}{4\pi} \epsilon^{\mu\nu\lambda} \, A_\mu \underline{t}^T \partial_\nu \underline{a}_\lambda \, . \tag{9}$$

In Eq. 9, $\underline{a}_\mu = (a_{1,\mu}, \cdots, a_{n,\mu})^T$ is the n-component auxiliary compact $U(1)$ gauge field, $\underline{\underline{K}}$ is a symmetric $n-$by$-n$ integer matrix, $\underline{t}$ is a charge vector, and e is the charge of the external gauge field A_μ.

(2.a) For the classical Lagrangian $\mathcal{L}$, derive the Euler-Lagrange equations. Show that the A_μ current operator $J_\mu = \frac{e}{2\pi} \sum_i \epsilon^{\mu\nu\lambda} \, \partial_\nu a_{i,\lambda}$ is quantized according to $J_\mu = \mathcal{C} \frac{e^2}{2\pi} \epsilon^{\mu\nu\lambda} \, \partial_\nu A_\lambda$, where the many-body Chern number is given by

$$\mathcal{C} = \sum_{i,j} t_i \big(K^{-1}\big)_{ij} \, . \tag{10}$$

(2.b) Here take $n = 1$. Integrate out the auxiliary $U(1)$ gauge field and show that the effective Lagrangian for A_μ is of the form of a Chern-Simons gauge field.

(2.c) Show that the Chern-Simons term is gauge invariant (up to surface terms).