Sheet 10:

Hand-out: Tuesday, Jan. 09, 2024; Solutions: Tuesday, Jan. 16, 2024

Problem 1 Girvin-Macdonald-Platzman theory of neutral excitations

Consider an incompressible ground state in the lowest Landau level (LLL) with homogeneous density ρ_0 and energy E_0 which is described by the wavefunctions $\Psi(\{\mathbf{r}_j\})$ – e.g. the $1/m$ Laughlin state. We use the density operator, which reads in first quantization

$$\hat{\delta}(\mathbf{R}) = \sum_{j=1}^{N} \delta(\mathbf{R} - \hat{\mathbf{r}}_j) \tag{1}$$

for particles $j = 1, .., N$, to construct a corresponding trial state for neutral excitations [single mode approximation (SMA)]:

$$\Phi_{\mathbf{k}}(\{\mathbf{r}_j\}) = \frac{1}{\sqrt{N}} \hat{\rho}_{\mathbf{k}} \Psi(\{\mathbf{r}_j\}), \tag{2}$$

where

$$\hat{\rho}_{\mathbf{k}} = \int d^2\mathbf{R} e^{-i\mathbf{k}\cdot\mathbf{R}} \hat{\delta}(\mathbf{R}) = \sum_{j=1}^{N} e^{-i\mathbf{k}\cdot\hat{\mathbf{r}}_j}. \tag{3}$$

(1.a) Show that $\langle \Phi_{\mathbf{k}} | \Psi \rangle = 0$ for all $\mathbf{k} \neq 0$.

(1.b) Generalize the ansatz (**??**) to obtain a trial state in the LLL, and show that

$$\bar{\rho}_{\mathbf{k}} = \hat{\mathcal{P}}_{\rm LLL} \hat{\rho}_{\mathbf{k}} \hat{\mathcal{P}}_{\rm LLL} = \sum_{j=1}^{N} e^{-ik\partial_{z_j}} e^{-\frac{i}{2}k^* z_j}, \tag{4}$$

where $z_j = x_j + iy_j$ and $k = k_x + ik_y$

(1.c) Show that the excitation energy of the LLL-projected SMA ansatz $\bar{\Phi}_{\mathbf{k}} = \hat{\mathcal{P}}_{\rm LLL} \Phi_{\mathbf{k}}$ is:

$$\Delta_{\mathbf{k}} = \frac{\langle \bar{\Phi}_{\mathbf{k}} | \hat{\mathcal{H}} - E_0 | \bar{\Phi}_{\mathbf{k}} \rangle}{\langle \bar{\Phi}_{\mathbf{k}} | \bar{\Phi}_{\mathbf{k}} \rangle} \equiv \frac{\bar{f}(\mathbf{k})}{\bar{s}(\mathbf{k})} \tag{5}$$

with $E_0 = \frac{\langle \Psi | \hat{\mathcal{H}} | \Psi \rangle}{\langle \Psi | \Psi \rangle}$.

(1.d) Derive the following properties of the projected density operator:

$$\bar{\rho}^{\dagger}_{\mathbf{k}} = \bar{\rho}_{-\mathbf{k}} \tag{6a}$$

$$[\bar{\rho}_{\mathbf{k}}, \bar{\rho}_{\mathbf{q}}] = [e^{k^* q/2} - e^{kq^*/2}] \bar{\rho}_{\mathbf{k}+\mathbf{q}} \tag{6b}$$

$$\hat{\mathcal{P}}_{\rm LLL} \hat{\rho}^{\dagger}_{\mathbf{k}} \hat{\rho}_{\mathbf{k}} = \bar{\rho}^{\dagger}_{\mathbf{k}} \bar{\rho}_{\mathbf{k}} + (1 - e^{qq^*/2}) \tag{6c}$$

(1.e) Show that $\bar{s}(\mathbf{k}) = s(\mathbf{k}) - (1 - e^{-|k|^2/2})$, where $s(\mathbf{k}) = \frac{1}{N}\langle\Psi|\hat{\rho}^\dagger_{\mathbf{k}}\hat{\rho}_{\mathbf{k}}|\Psi\rangle$ is the static structure factor.

(1.f) Show that

$$\bar{f}(\mathbf{k}) = \frac{1}{N}\langle\Psi|\bar{\rho}^\dagger_{\mathbf{k}}[\hat{\mathcal{P}}_{\mathrm{LLL}}\hat{\mathcal{H}}_{\mathrm{int}}\hat{\mathcal{P}}_{\mathrm{LLL}}, \bar{\rho}_{\mathbf{k}}]|\Psi\rangle \tag{7}$$

and write

$$\hat{\mathcal{P}}_{\mathrm{LLL}}\hat{\mathcal{H}}_{\mathrm{int}}\hat{\mathcal{P}}_{\mathrm{LLL}} = \int \frac{d^2\mathbf{q}}{2\pi^2} V(\mathbf{q})\left(\bar{\rho}^\dagger_{\mathbf{q}}\bar{\rho}_{\mathbf{q}} - Ne^{-|q|^2/2}\right), \tag{8}$$

where $V(\mathbf{q})$ is the Fourier transform of the interaction potential $V(\mathbf{r}_i - \mathbf{r}_j)$.

(1.g) Show that

$$\bar{f}(\mathbf{k}) = \frac{1}{2}\sum_{\mathbf{q}} V(\mathbf{q})\left(e^{q^*k/2} - e^{qk^*/2}\right) \tag{9}$$

$$\times\left[\bar{S}(\mathbf{q})e^{-|k|^2/2}\left(e^{-qk^*/2} - e^{-q^*k/2}\right) + \bar{S}(\mathbf{k}+\mathbf{q})\left(e^{qk^*/2} - e^{q^*k/2}\right)\right] \tag{10}$$

Problem 2 Journal Club: Non-Abelian braiding

Read the Chapters I-II.C of the review article *Nayak et al., Rev. Mod. Phys. 80 (2008)*. Discuss the following questions:

- What is the concept of topological quantum computing?
- Why are topological qubits inherently robust against errors?
- Which of the FQH state is a promising candidate for quantum computing and why?

Hint: For a pedagogical introduction on non-Abelian fractional quantum Hall states, see Chapter 4 of David Tong's lectures on the quantum Hall effect ,`https://www.damtp.cam.ac.uk/user/tong/qhe/four.pdf`.