

# Quantum Field Theory (Quantum Electrodynamics)

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12 February 2020

First and last name : \_\_\_\_\_

Matriculation number : \_\_\_\_\_

Number of extra sheets : \_\_\_\_\_

### Guidelines :

- The exam consists of 4 problems.
- The duration of the exam is 3 hours.
- Please write your name and matriculation number on every sheet that you hand in. State the number of extra sheets.
- You are not allowed to use books or notes. Some potentially useful formulas can be found on the last page.
- Do you agree that your results be published on the course website?  yes  no
- Your answers should be comprehensible and readable.

GOOD LUCK!

Do not write below this line.

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Comments :

Exercise 1	/ 16 P
Exercise 2	/ 40 P
Exercise 3	/ 22 P
Exercise 4	/ 22 P

Total	/ 100 P
Grade	

## Problem 1 (16 points)

Consider a column of  $N$  real scalar fields

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_N(x) \end{pmatrix}$$

1. Construct the most general Lagrangian (in four spacetime dimensions) which is Lorentz and  $O(N)$  invariant and contains terms with mass dimension at most 4 in  $\Phi$  and its derivatives.

*Hint : The  $O(N)$  transformation acts on the fields as  $\phi'_i = \sum_j O_{ij} \phi_j$ , with  $O_{ij}$  a (constant) real orthogonal matrix ( $O^T O = 1$ ).*

2. Find the equations of motion.
3. In its infinitesimal form, an  $O(N)$  transformation can be written as

$$O_{ij} = \delta_{ij} + \sum_A \epsilon_A T_{ij}^A + \mathcal{O}(\epsilon^2) ,$$

where  $\epsilon^A \ll 1$  are the (constant) parameters of the transformation,  $T_{ij}^A$  are the so-called generators of  $O(N)$ , and  $A$  runs over the number of independent generators. Show that the  $T_{ij}^A$  are antisymmetric matrices.

4. Find the Noether current associated with the  $O(N)$  invariance of the theory. Show that it is conserved on the equations of motion.
5. Derive the Hamiltonian of this theory.

## Problem 2 (40 points)

Consider the following Lagrangian (in four spacetime dimensions)

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\not{\partial} - m)\psi - g\phi\bar{\psi}\gamma_5\psi .$$

Here  $\phi$  is a real scalar field with mass  $M$ ,  $\psi$  the electron-positron field with mass  $m$ ,  $g$  a coupling constant, and  $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ .

1. Check if the Lagrangian is invariant under  $\psi \rightarrow e^{i\alpha}\psi$ , with  $\alpha$  a constant. If so, derive the corresponding Noether current.
2. Consider the process

$$e^- e^+ \rightarrow e^- e^+ .$$

Draw and label the leading-order Feynman diagram or diagrams. Please use  $p_1, p_2$  to indicate the incoming momenta, and  $p_3, p_4$  the outgoing momenta.

3. Derive the spin-averaged amplitude squared in terms of the Mandelstam variables, in the limit  $m \rightarrow 0$ .

4. Using the above result, derive the spin-averaged amplitude squared for the process

$$e^- e^- \rightarrow e^- e^- .$$

*Hint : You should not need to do any explicit computation.*

5. Consider the process  $\phi \rightarrow e^+ e^-$ . For what masses can this process take place?

### Problem 3 (22 points)

1. Write down the QED Lagrangian.
2. Derive the equations of motion for the fields.
3. Write down the Feynman diagrams for the process  $e^- \gamma \rightarrow e^- \gamma$  and show that the corresponding amplitude can be written as

$$i\mathcal{M} = \epsilon^\mu(\vec{k}_1) \epsilon^\nu(\vec{k}_2) \mathcal{A}_{\mu\nu} .$$

*Hint : You don't need to fully simplify your result.*

4. Show that  $k_1^\mu \mathcal{A}_{\mu\nu} = 0$  and explain why this is expected.

*Hint : Use four-momentum conservation and Dirac equation in momentum space.*

### Problem 4 (22 points)

1. Using a complex scalar field  $\phi$  with mass  $m_\phi$  and the photon  $A_\mu$ , write down the most general Lorentz and  $U(1)$  gauge invariant Lagrangian up to (and including) terms of mass dimension 4.
2. Draw and label the two one-loop Feynman diagrams associated with the photon vacuum polarization in this theory.
3. Now work in four spacetime dimensions and check that  $q^\mu \Pi_{\mu\nu}(q) = 0$ .

*Hint : You do not need to explicitly carry out the integrals. Use the relation  $\pm 2qk + q^2 = (k \pm q)^2 - m^2 - (k^2 - m^2)$ .*

# Potentially useful formulas

## Decomposition in terms of ladder operators

### Massive real scalar field

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} (a(\vec{p})e^{-ipx} + a^\dagger(\vec{p})e^{ipx}) , \quad \omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2} .$$

$$[a(\vec{p}), a^\dagger(\vec{p}')] = (2\pi)^3 2\omega_{\vec{p}} \delta^{(3)}(\vec{p} - \vec{p}') .$$

### Massive complex scalar field

$$\chi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} (a(\vec{p})e^{-ipx} + b^\dagger(\vec{p})e^{ipx}) , \quad \omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2} .$$

$$[a(\vec{p}), a^\dagger(\vec{p}')] = [b(\vec{p}), b^\dagger(\vec{p}')] = (2\pi)^3 2\omega_{\vec{p}} \delta^{(3)}(\vec{p} - \vec{p}') .$$

### Massive fermionic field

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_{\vec{p}}} \sum_i (u_i(\vec{p})a_i(\vec{p})e^{-ipx} + v_i(\vec{p})b_i^\dagger(\vec{p})e^{ipx}) , \quad \omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2} .$$

$$\{a_i(\vec{p}), a_j^\dagger(\vec{p}')\} = \{b_i(\vec{p}), b_j^\dagger(\vec{p}')\} = (2\pi)^3 2\omega_{\vec{p}} \delta_{ij} \delta^{(3)}(\vec{p} - \vec{p}') .$$

$$(\not{p} - m)u(p) = 0 , \quad \bar{u}(p)(\not{p} - m) = 0 .$$

$$(\not{p} + m)v(p) = 0 , \quad \bar{v}(p)(\not{p} + m) = 0 .$$

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m , \quad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m .$$

### Photon

$$A^\mu(x) = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \sum_{r=0}^3 \epsilon_r^\mu \left( a_r(\vec{k})e^{-ikx} + a_r^\dagger(\vec{k})e^{ikx} \right) , \quad \omega_{\vec{k}} = |\vec{k}| .$$

$$[a_r(\vec{k}), a_s^\dagger(\vec{k}')] = (2\pi)^2 2\omega_{\vec{k}} \zeta_s \delta_{rs} \delta^{(3)}(\vec{k} - \vec{k}') , \quad \zeta_0 = -1, \zeta_{1,2,3} = 1 .$$

$$\eta_{\mu\nu} \epsilon_r^\mu \epsilon_s^\nu = -\zeta_s \delta_{rs} , \quad \sum_{r=0}^3 \zeta_r \epsilon_r^\mu \epsilon_r^\nu = -\eta^{\mu\nu} .$$

## $\gamma$ matrices properties

$$\begin{aligned}\{\gamma^\mu, \gamma^\nu\} &= 2\eta^{\mu\nu}, \quad \gamma^\mu \gamma_\mu = 4, \quad \gamma^\mu \gamma^\alpha \gamma_\mu = -2\gamma^\alpha, \quad \gamma^\mu \gamma^\alpha \gamma^\beta \gamma_\mu = 4\eta^{\alpha\beta}, \\ \gamma^\mu \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma_\mu &= -2\gamma^\gamma \gamma^\beta \gamma^\alpha, \\ (\gamma_5)^2 &= 1, \quad \gamma_5^+ = \gamma_5, \quad \text{with } \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3, \\ \{\gamma_5, \gamma_\mu\} &= 0, \\ \text{tr}(\gamma^\mu \gamma^\nu) &= 4\eta^{\mu\nu}, \\ \text{tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta) &= 4(\eta^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\alpha}\eta^{\nu\beta}). \\ \text{tr}(\gamma_5) &= 0.\end{aligned}$$

## Projection Operators

$$P_{L,R} = \frac{1 \mp \gamma_5}{2},$$

$$P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = 0, \quad P_R P_L = 0, \quad P_L + P_R = 1$$

## Mandelstam variables

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \quad t = (p_1 - p_3)^2 = (p_2 - p_4)^2, \quad u = (p_1 - p_4)^2 = (p_2 - p_3)^2.$$

## Feynman rules for QED

$$\begin{array}{l}
 \longrightarrow \longrightarrow \longrightarrow \rightarrow \left( \frac{i}{\not{p} - m + i\varepsilon} \right) \\
 \mu \text{ ~~~~~ } \nu \rightarrow \frac{-i\eta_{\mu\nu}}{p^2 + i\varepsilon} \\
 \begin{array}{c} \nearrow \\ \searrow \end{array} \text{ ~~~~~ } \mu \rightarrow -ie\gamma^\mu \\
 \text{Incoming fermion: } \longrightarrow \bullet \rightarrow u(\vec{p}, s) \\
 \text{Incoming antifermion: } \longleftarrow \bullet \rightarrow \bar{v}(\vec{p}, s) \\
 \text{Outgoing fermion: } \bullet \longrightarrow \rightarrow \bar{u}(\vec{p}, s) \\
 \text{Outgoing antifermion: } \bullet \longleftarrow \rightarrow v(p, s) \\
 \text{photon: } \mu \text{ ~~~~~ } \bullet \rightarrow \epsilon_\mu(\vec{k}, \lambda)
 \end{array}$$

## Cubic and quartic vertices for scalar - photon interaction

$$\begin{array}{l}
 \begin{array}{c} \nearrow \\ \searrow \end{array} \text{ ~~~~~ } \mu \rightarrow ie(p + p')^\mu \\
 \begin{array}{c} \nearrow \\ \searrow \end{array} \text{ ~~~~~ } \nu \rightarrow 2ie^2\eta^{\mu\nu}
 \end{array}$$