Problem 1 (16 points)

Consider a column of N real scalar fields

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_N(x) \end{pmatrix}$$

1. Construct the most general Lagrangian (in four spacetime dimensions) which is Lorentz and $O(N)$ invariant and contains terms with mass dimension at most 4 in Φ and its derivatives.
   Hint : The $O(N)$ transformation acts on the fields as $\phi'_i = \sum_j O_{ij}\phi_j$, with O_{ij} a (constant) real orthogonal matrix ($O^T O = 1$).
2. Find the equations of motion.
3. In its infinitesimal form, an $O(N)$ transformation can be written as

   $$O_{ij} = \delta_{ij} + \sum_A \epsilon_A T^A_{ij} + \mathcal{O}(\epsilon^2) \ ,$$

   where $\epsilon^A \ll 1$ are the (constant) parameters of the transformation, T^A_{ij} are the so-called generators of $O(N)$, and A runs over the number of independent generators. Show that the T^A_{ij} are antisymmetric matrices.
4. Find the Noether current associated with the $O(N)$ invariance of the theory. Show that it is conserved on the equations of motion.
5. Derive the Hamiltonian of this theory.

1.

   $$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^T \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^T \Phi - \frac{\lambda}{4}(\Phi^T\Phi)^2 \ ,$$

   with $\lambda > 0$.

   3 points : 1 for each term (-0.5 points for wrong relative factors/signs)

2.

   $$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) - \frac{\partial \mathcal{L}}{\partial \Phi} = 0$$

   $$(\Box + m^2)\Phi = -\lambda(\Phi^T\Phi)\Phi$$

   3 points : 1 for Euler-Lagrange equations, 1 for $\Box + m^2$, 1 for λ

3. From

   $$\mathbb{1} = O^T O = \left(\mathbb{1} + \epsilon_A T^A + \mathcal{O}(\epsilon^2)\right)\left(\mathbb{1} + \epsilon_A (T^A)^T + \mathcal{O}(\epsilon^2)\right) = \mathbb{1} + \epsilon_A \left(T^A + (T^A)^T\right) + \mathcal{O}(\epsilon^2),$$

   follows

   $$T^A = -(T^A)^T \quad \forall\ \epsilon_A.$$

   2 points : 1 for the correct formula, 1 for the end-result

4.

$$\Phi \to \Phi' = \Phi + \delta\Phi = \Phi + \epsilon_A T^A \Phi + \mathcal{O}(\epsilon^2)$$

$$j^A_\mu \epsilon_A = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \Phi)} \delta\Phi = (\partial_\mu \Phi)^T T^A \Phi \epsilon_A$$

$$\partial^\mu j^A_\mu = \Box \Phi^T T^A \Phi + (\partial_\mu \Phi)^T T^A \partial^\mu \Phi = -(m^2 + \lambda \Phi^T \Phi)\Phi^T T^A \Phi + (\partial_\mu \Phi)^T T^A \partial^\mu \Phi = 0$$

each piece independently is zero, due to symmetric contracted with anti-symmetric.

6 points : 1 for $\delta\phi$, 1 for $\frac{\partial \mathcal{L}}{\partial(\partial\phi)} \delta\phi$, 1 for result, 1.5 for using eom, 1.5 for antisymmetry

5.

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \dot{\Phi}^T$$

$$H = \int d^3x \mathcal{H} = \int d^3x \left(\pi \dot{\Phi} - \mathcal{L}\right) = \int d^3x \left(\frac{1}{2}\pi\pi^T + \frac{1}{2}(\partial_i \Phi)^T \partial_i \Phi + \frac{1}{2} m^2 \Phi^T \Phi + \frac{\lambda}{4}(\Phi^T \Phi)^2\right)$$

2 points : 1 for the definition of momenta and Hamiltonian (function of ϕ and π), 1 for result

Problem 2 (40 points)

Consider the following Lagrangian (in four spacetime dimensions)

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - g\phi\bar{\psi}\gamma_5\psi \ .$$

Here ϕ is a real scalar field with mass M, ψ the electron-positron field with mass m, g a coupling constant, and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$.

1. Check if the Lagrangian is invariant under $\psi \to e^{i\alpha}\psi$, with α a constant. If so, derive the corresponding Noether current.
2. Consider the process
$$e^- e^+ \to e^- e^+ \ .$$
Draw and label the leading-order Feynman diagram or diagrams. Please use p_1, p_2 to indicate the incoming momenta, and p_3, p_4 the outgoing momenta.
3. Derive the spin-averaged amplitude squared in terms of the Mandelstam variables, in the limit $m \to 0$.
4. Using the above result, derive the spin-averaged amplitude squared for the process
$$e^- e^- \to e^- e^- \ .$$
Hint : You should not need to do any explicit computation.
5. Consider the process $\phi \to e^+ e^-$. For what masses can this process take place?

1. $\psi \to e^{i\alpha}\psi$ and $\bar{\psi} \to e^{-i\alpha}\bar{\psi}$ since α is a constant real number
   (a) $\bar{\psi}\psi \to \bar{\psi}\psi$
   (b) $\bar{\psi}\partial\!\!\!/\psi \to \bar{\psi}\partial\!\!\!/\psi$
   (c) $\bar{\psi}\gamma_5\psi \to \bar{\psi}\gamma_5\psi$

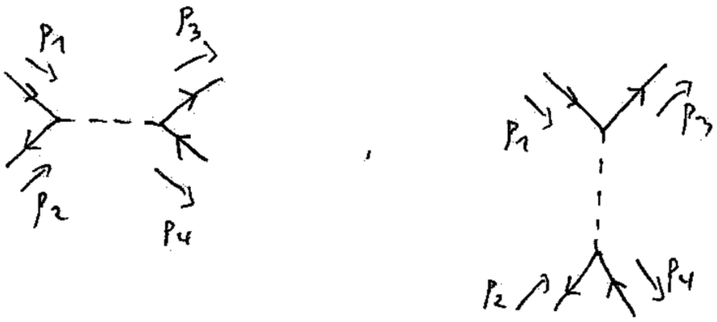
so the Lagrangian is invariant, we get $\psi \to \psi + i\alpha\psi + \mathcal{O}(\alpha^2)$ and $\bar{\psi} \to \bar{\psi} - i\alpha\bar{\psi} + \mathcal{O}(\alpha^2)$

$$j^\mu \alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} i\psi\alpha = -\bar{\psi}\gamma^\mu\psi\alpha$$

since the Lagrangian does not depend on $\partial\bar{\psi}$.

5 points : 3 for each fermion term, 2 points for current

2. Figures :



6 points : 3 for each diagram (2 for drawing, 1 for labeling)

3. 21 points :

$$i\mathcal{M} = \bar{v}(\vec{p}_2, s_2)(-ig)\gamma_5 u(\vec{p}_1, s_1)\frac{i}{(P_1+P_2)^2 - M^2}\bar{u}(\vec{p}_3, s_3)(-ig)\gamma_5 v(\vec{p}_4, s_4) -$$

$$\bar{u}(\vec{p}_3, s_3)(-ig)\gamma_5 u(\vec{p}_1, s_1)\frac{i}{(P_1-P_3)^2 - M^2}\bar{v}(\vec{p}_2, s_2)(-ig)\gamma_5 v(\vec{p}_4, s_4) = -ig^2(\mathcal{M}_s - \mathcal{M}_t)$$

2+1 (1 for the correct vertex) points for each part + 1 point for the correct relative sign

using Mandelstam variables, denoting $u(\vec{p}_j, s_j) = u_j$, $v(\vec{p}_j, s_j) = v_j$ and averaging amplitude

$$\sum_s \frac{1}{4}|\mathcal{M}|^2 = \sum_s \frac{g^4}{4}(|\mathcal{M}_s|^2 + |\mathcal{M}_t|^2 - (\mathcal{M}_s\mathcal{M}_t^* + \mathcal{M}_t\mathcal{M}_s^*))$$

1 point for the correct formula above

$$\sum_s |\mathcal{M}_s|^2 = \frac{1}{(s-M^2)^2}\sum_s tr(\gamma_5 v_2\bar{v}_2\gamma_5 u_1\bar{u}_1)tr(\gamma_5 u_3\bar{u}_3\gamma_5 v_4\bar{v}_4) =$$

$$= \frac{1}{(s-M^2)^2} tr[\gamma_5 \not{P}_2\gamma_5\not{P}_1]tr[\gamma_5\not{P}_3\gamma_5\not{P}_4] = \frac{(-1)^2}{(s-M^2)^2}16(P_1P_2)(P_3P_4)$$

3 points for the correct computation (2 points for trace, 1 for spin averaging)

$$\sum_s |\mathcal{M}_t|^2 = \frac{1}{(t-M^2)^2}\sum_s tr(\gamma_5 u_3\bar{u}_3\gamma_5 u_1\bar{u}_1)tr(\gamma_5 v_2\bar{v}_2\gamma_5 v_4\bar{v}_4) =$$

$$= \frac{1}{(t-M^2)^2} tr[\gamma_5 \not{P}_3\gamma_5\not{P}_1]tr[\gamma_5\not{P}_2\gamma_5\not{P}_4] = \frac{(-1)^2}{(t-M^2)^2}16(P_1P_3)(P_2P_4)$$

3 points for the correct computation (2 points for trace, 1 for spin averaging)

$$\sum_s \mathcal{M}_s\mathcal{M}_t^* = \frac{1}{(t-M^2)(s-M^2)}\sum_s tr(v_4\bar{v}_4\gamma_5 v_2\bar{v}_2\gamma_5 u_1\bar{u}_1\gamma_5 u_3\bar{u}_3\gamma_5) =$$

$$= \frac{1}{(t-M^2)(s-M^2)}\sum_s tr(\not{P}_4\gamma_5\not{P}_2\gamma_5\not{P}_1\gamma_5\not{P}_3\gamma_5) =$$

$$= \frac{(-1)^2\cdot 4}{(t-M^2)(s-M^2)}((P_1P_2)(P_3P_4)+(P_2P_4)(P_1P_3)-(P_2P_3)(P_1P_4)) = \sum_s \mathcal{M}_t\mathcal{M}_s^*$$

3 points for both the above terms (1.5 each)
For showing explicitly the γ_5 manipulations (changing the sign of M^* and correct sign when anticommuting γ_5) we give 1 + 1 points, in total

$$(P_1P_3) = (P_2P_4) = -\frac{t}{2}$$
$$(P_1P_2) = (P_3P_4) = \frac{s}{2}$$
$$(P_1P_4) = (P_2P_3) = -\frac{u}{2}$$

1 point for the above

$$\sum_s \frac{1}{4}|\mathcal{M}|^2 = g^4\left[\frac{s^2}{(s-M^2)^2} + \frac{t^2}{(t-M)^2} - \frac{1}{2}\frac{s^2+t^2-u^2}{(s-M^2)(t-M^2)}\right],$$

or, using $u^2 = (s+t)^2$,

$$\sum_s \frac{1}{4}|\mathcal{M}|^2 = g^4\left[\frac{s^2}{(s-M^2)^2} + \frac{t^2}{(t-M)^2} + \frac{st}{(s-M^2)(t-M^2)}\right].$$

1 point for the final result

4. $|\mathcal{M}(e^-(P_1)e^-(P_2)\to e^-(P_3)e^-(P_4))|^2 = |\mathcal{M}(e^-(P_1)e^+(-P_4)\to e^-(P_3)e^+(-P_2))|^2$ which means $s\leftrightarrow u$ and $t\leftrightarrow t$

$$\sum_s \frac{1}{4}|\mathcal{M}|^2 = g^4\left[\frac{u^2}{(u-M^2)^2} + \frac{t^2}{(t-M)^2} + \frac{ut}{(u-M^2)(t-M^2)}\right]$$

5 points : 3 for exchanging momenta, 2 for Mandelstam

5. Energy-momentum conservation requires $M \geq 2m$. But for $M = 2m$ amplitude is identically zero, so we need $M > 2m$.
3 points : 2 for $\geq$, +1 for the stricter bound $>$ (including the explanation that the amplitude vanishes for $M = 2m$)

Problem 3 (22 points)

1. Write down the QED Lagrangian.
2. Derive the equations of motion for the fields.
3. Write down the Feynman diagrams for the process $e^-\gamma \to e^-\gamma$ and show that the corresponding amplitude can be written as

$$i\mathcal{M} = \epsilon^\mu(\vec{k}_1)\epsilon^\nu(\vec{k}_2)\mathcal{A}_{\mu\nu} \ .$$

Hint : You don't need to fully simplify your result.

4. Show that $k_1^\mu \mathcal{A}_{\mu\nu} = 0$ and explain why this is expected.
Hint : Use four-momentum conservation and Dirac equation in momentum space.

1.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}A\!\!\!/\psi$$

3 points : 1 for each term. If they use covariant derivative they should define and explain

2.

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0$$

$$\partial_\mu\left[-\frac{1}{2}F^{\alpha\beta}\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha\right)\right] + e\bar{\psi}\gamma^\nu\psi = 0$$

$$-\partial_\mu F^{\mu\nu} + e\bar{\psi}\gamma^\nu\psi = 0$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) - \frac{\partial\mathcal{L}}{\partial\bar{\psi}} = 0$$

$$(i\partial\!\!\!/ - m)\psi = eA\!\!\!/\psi$$

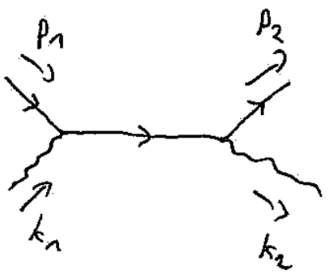
with same manner

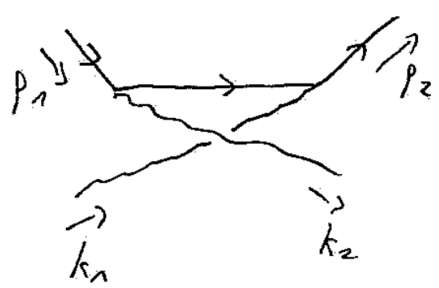
$$i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = -e\bar{\psi}A\!\!\!/$$

4 points : 1 for E-L, 2 for photon (1 for explicit calculation), 1 for fermions

3. 7 points :

Figures :





2 points for each diagram (1 for drawing, 1 for labeling)

$$i\mathcal{M} = \bar{u}_2(-ie)\gamma^\mu \frac{i}{\not{P}_1 + \not{k}_1 - m}(-ie)\gamma^\nu u_1 \epsilon_\mu(\vec{k}_2)\epsilon_\nu(\vec{k}_1) +$$
$$+ \bar{u}_2(-ie)\gamma^\mu \frac{i}{\not{P}_1 - \not{k}_2 - m}(-ie)\gamma^\nu u_1 \epsilon_\mu(\vec{k}_1)\epsilon_\nu(\vec{k}_2)$$

2 points, 1 for each term

$i\mathcal{M} = \epsilon_\mu(\vec{k}_1)\epsilon_\nu(\vec{k}_2)A^{\mu\nu}$, with

$$A^{\mu\nu} = -ie^2\left[\bar{u}_2\gamma^\nu \frac{1}{\not{P}_1 + \not{k}_1 - m}\gamma^\mu u_1 + \bar{u}_2\gamma^\mu \frac{1}{\not{P}_1 - \not{k}_2 - m}\gamma^\nu u_1\right]$$

1 point for writing $A^{\mu\nu}$ explicitly

4. 8 points :

$$k_{1\mu}A^{\mu\nu} = -ie^2\left[\bar{u}_2\gamma^\nu \frac{1}{\not{P}_1 + \not{k}_1 - m}\not{k}_1 u_1 + \bar{u}_2\not{k}_1 \frac{1}{\not{P}_1 - \not{k}_2 - m}\gamma^\nu u_1\right]$$

Using $(\not{P}_1 - m)u_1 = 0$ and $\bar{u}_2(\not{P}_2 - m) = 0$, we get

$$\frac{1}{\not{P}_1 + \not{k}_1 - m}\not{k}_1 u_1 = \frac{1}{\not{P}_1 + \not{k}_1 - m}(\not{k}_1 + \not{P}_1 - m)u_1 = u_1$$

and

$$\bar{u}_2\not{k}_1 \frac{1}{\not{P}_1 - \not{k}_2 - m} = \bar{u}_2(\not{k}_1 - \not{P}_2 + m)\frac{1}{\not{P}_2 - \not{k}_1 - m} = -\bar{u}_2,$$

2 points for each term for using Dirac equation + 2 points in total for momentum conservation

so we get

$$k_{1\mu}A^{\mu\nu} \propto \bar{u}_2\gamma^\nu u_1 - \bar{u}_2\gamma^\nu u_1 = 0.$$

1 point for the final result

This result shows, that on physical amplitudes longitudinal photon vanishes.

1 point for explanation (simply writing Ward identity or current conservation is not enough)

Problem 4 (22 points)

1. Using a complex scalar field ϕ with mass m_ϕ and the photon A_μ, write down the most general Lorentz and $U(1)$ gauge invariant Lagrangian up to (and including) terms of mass dimension 4.
2. Draw and label the two one-loop Feynman diagrams associated with the photon vacuum polarization in this theory.

3. Now work in four spacetime dimensions and check that $q^\mu \Pi_{\mu\nu}(q) = 0$.
Hint : You do not need to explicitly carry out the integrals. Use the relation $\pm 2qk + q^2 = (k \pm q)^2 - m^2 - (k^2 - m^2)$.

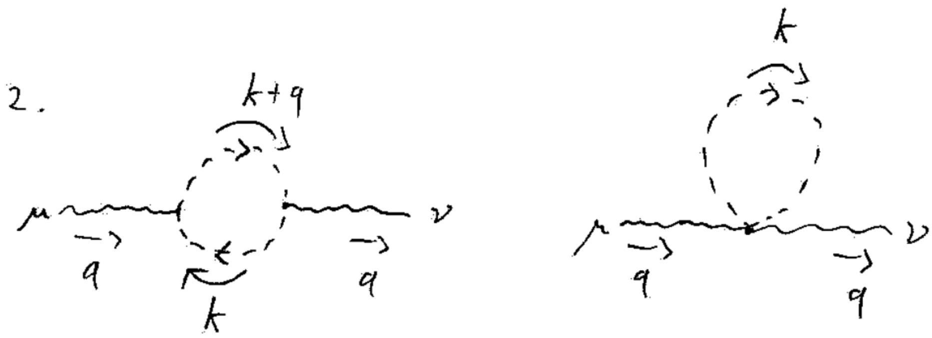
1.
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - m_\phi^2 \phi^* \phi - \frac{\lambda}{4} (\phi^* \phi)^2$$

with $D_\mu = \partial_\mu - ieA_\mu$, since under $\phi \to e^{i\alpha(x)}\phi$, $\phi^*\phi$ is invariant and $D_\mu\phi \to e^{i\alpha} D_\mu \phi$, which makes kinetic term invariant as well. One could also expand the scalar kinetic term directly :

$$(D_\mu \phi)^* D^\mu \phi = (\partial_\mu \phi)^* \partial^\mu \phi - ieA^\mu \left(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi\right) + e^2 A_\mu A^\mu \phi \phi^*$$

6 points total : 1 for each term

2. Figure :



6 points : 3 for each diagram (2 for drawing, 1 for labeling)

3. 10 points :

$$i\Pi(q)_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} (ie)(2k_\mu + q_\mu) \frac{i}{(k+q)^2 - m_\phi^2} (ie)(2k_\nu + q_\nu) \frac{i}{k^2 - m_\phi^2} + \int \frac{d^4k}{(2\pi)^4} 2ie^2 \eta_{\mu\nu} \frac{i}{k^2 - m_\phi^2}$$

4 points, 2 for each term

$$iq^\mu \Pi(q)_{\mu\nu} = e^2 \int \frac{d^4k}{(2\pi)^4} (2kq + q^2) \frac{1}{(k+q)^2 - m_\phi^2} (2k_\nu + q_\nu) \frac{1}{k^2 - m_\phi^2} - 2e^2 q_\nu \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_\phi^2}.$$

Using the hint, we get

$$(2kq + q^2) \frac{1}{(k+q)^2 - m_\phi^2} (2k_\nu + q_\nu) \frac{1}{k^2 - m_\phi^2} = (2k_\nu + q_\nu) \frac{(k+q)^2 - m_\phi^2 - (k^2 - m_\phi^2)}{[(k+q)^2 - m_\phi^2][k^2 - m_\phi^2]} =$$
$$= (2k_\nu + q_\nu) \left[\frac{1}{k^2 - m_\phi^2} - \frac{1}{(k+q)^2 - m_\phi^2} \right],$$

1 point for using the hint correctly

which gives

$$iq^\mu \Pi(q)_{\mu\nu} = e^2 \int \frac{d^4k}{(2\pi)^4}(2k_\nu + q_\nu)\left[\frac{1}{k^2 - m_\phi^2} - \frac{1}{(k+q)^2 - m_\phi^2}\right] - 2e^2 q_\nu \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - m_\phi^2}.$$

In the second term, substituting $k + q = l$, we get

$$iq^\mu \Pi(q)_{\mu\nu} = e^2 \int \frac{d^4k}{(2\pi)^4}\frac{2k_\nu + q_\nu}{k^2 - m_\phi^2} - e^2 \int \frac{d^4l}{(2\pi)^4}\frac{2l_\nu - q_\nu}{l^2 - m_\phi^2} - 2e^2 q_\nu \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - m_\phi^2}.$$

3 points for substitution

Relabeling $l \to k$, we get $q^\mu \Pi(q)_{\mu\nu} = 0$.

2 points for end-result