Sheet 14.2: Complex Analysis

Posted: Mo 30.01.23 Central Tutorial: Th 02.02.23 Due: Th 09.02.23, 14:00
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 1, 2(a,b), 4(a).
Videos exist for example problems 2 (C9.4.1), 4 (C9.4.7).

Example Problem 1: Cauchy's theorem [3]
Points: (a)[1](E); (b)[2](E).

The function $f(z) = \mathrm{e}^z$ for $z \in \mathbb{C}$ is analytic. Cauchy's theorem then states that (a) closed contour integrals over simply connected domains are zero, and (b) contour integrals between two points are independent of the chosen contour. Check these claims explicitly by calculating the following contour integrals:

(a) $I_{\gamma_R} = \oint_{\gamma_R} \mathrm{d}z\, f(z)$, along the circle γ_R with radius R about the origin $z = 0$.

(b) $I_{\gamma_i} = \int_{\gamma_i} \mathrm{d}z\, f(z)$, between the points $z_0 = 0$ and $z_1 = 1 - \mathrm{i}$, along (i) the straight line $\gamma_1 \colon z(t) = (1-\mathrm{i})t$, and (ii) the curve $\gamma_2 \colon z(t) = t^3 - \mathrm{i}t$, with $t \in (0,1)$. Calculate explicitly the difference $F(z_1) - F(z_0)$, where $F(z)$ is the antiderivative of $f(z)$.

Example Problem 2: Circular contours, residue theorem [4]
Points: (a)[1](E); (b)[1](M); (c)[2](A)

(a) Calculate the integrals $I_+^{(k)} = \oint_{k \text{ times: } |z|=R} \frac{\mathrm{d}z}{z}$ and $I_-^{(k)} = \oint_{k \text{ times: } |z|=R} \frac{\mathrm{d}z}{z}$, where $I_+^{(k)}$ (resp. $I_-^{(k)}$) involves a circular contour with radius R, winding around the origin k times in the mathematically positive (negative) direction, i.e. anticlockwise (clockwise). Do not use the residue theorem; rather calculate the integral directly using the parametrization $z(\phi) = R\,\mathrm{e}^{\mathrm{i}\phi}$ and a suitable choice of integration interval for ϕ.

Use the residue theorem to calculate the following closed contour integrals in the complex plane, for $0 < a \in \mathbb{R}$:

(b) $$I_1(a) = \oint_{|z|=\frac{1}{2}} \mathrm{d}z\, g(z)\,, \qquad I_2(a) = \oint_{2 \text{ times: } |z|=2} \mathrm{d}z\, g(z)\,, \quad \text{with} \quad g(z) = \frac{\mathrm{e}^{\mathrm{i}az}}{z^2+1}\,.$$

(c) $$I_3(a) = \oint_{|z|=4} \mathrm{d}z\, f(z)\,, \qquad \text{with} \quad f(z) = \frac{z}{z^3 + (a\mathrm{i}-6)z^2 + (9-6a\mathrm{i})z + 9a\mathrm{i}}\,.$$

Hint: One of the poles of $f(z)$ is at $z_1 = -a\mathrm{i}$.

[Check your results: (b) $I_2(\ln 2) = 3\pi$, (c) $I_3(1) = 0$, $I_3(6) = \frac{4\pi}{25}(1 + \frac{4}{3}\mathrm{i})$.]

Example Problem 3: Integration by contour closure and residue theorem [3]
Points: [3](A).

Calculate the following integral, with $a, b \in \mathbb{R}$, by closing the contour along a suitably chosen semicircle with radius $\to \infty$:

$$I(a,b) = \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{x^2 - 2xa + a^2 + b^2} . \qquad \text{[Check your results: } I(-1,-2) = \tfrac{\pi}{2}.\text{]}$$

Example Problem 4: Inverse Fourier transform via contour closure [4]
Points: (a)[2.5](A); (b)[1.5](A)

(a) The Green function defined by the equation $(\mathrm{d}_t + a)G(t) = \delta(t)$ (with $0 < a \in \mathbb{R}$) has a corresponding Fourier transform given by $\tilde{\mathcal{G}}(\omega) = (a - \mathrm{i}\omega)^{-1}$ (see sheet 12, problem 3). Show that the corresponding inverse Fourier transform yields the following result:

$$G(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{a - \mathrm{i}\omega} = \Theta(t)\,\mathrm{e}^{-at} , \quad \text{with} \quad \Theta(t) = \begin{cases} 1 & \text{for } t > 0 , \\ 0 & \text{for } t < 0 . \end{cases}$$

(b) The Fourier transform of the exponential function, $\tilde{L}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{e}^{\mathrm{i}\omega t} \mathrm{e}^{-a|t|} = \frac{2a}{\omega^2 + a^2}$ (with $0 < a \in \mathbb{R}$), is a Lorentz curve. Find the inverse Fourier transform $L(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega t} \tilde{L}(\omega)$, by explicitly calculating the integral.

Hint: Calculate the integral for $t \neq 0$ as a contour integral, by closing the contour with a suitably chosen semicircle with radius $\to \infty$.

[Total Points for Example Problems: 14]

Homework Problem 1: Cauchy's theorem [3]
Points: (a)[1](E); (b)[2](E).

Compute the contour integral $I_{\gamma_i} = \int_{\gamma_i} \mathrm{d}z \, (z - \mathrm{i})^2$ explicitly along the following contours, γ_i, and explain your answers with reference to Cauchy's theorem.

(a) γ_1 is the straight line from $z_0 = 0$ to $z_1 = 1$, γ_2 is the line from $z_1 = 1$ to $z_2 = \mathrm{i}$, and γ_3 is the line from $z_2 = \mathrm{i}$ to $z_0 = 0$. What is $I_{\gamma_1} + I_{\gamma_2} + I_{\gamma_3}$? Explain your answer.

(b) γ_4 is the quarter-circle with radius 1 from z_1 to z_2. Is there a connection between I_{γ_4} and the integrals from (a)?

Homework Problem 2: Circular contours, residue theorem [3]
Points: (a)[1.5](M); (b)[0.5](E); (c)[0.5](E); (d)[0.5](E).

Consider the function $f(z) = \dfrac{4z}{(z-a)(z+1)^2}$, with $1 < a \in \mathbb{R}$.

(a) Determine the residues of the function f at each of its poles.

Calculate the integral $I_{\gamma_i}(a) = \int_{\gamma_i} \mathrm{d}z \, f(z)$ for the following integration contours:

(b) γ_1: a circle with radius $R = 1$ about $z = a$, traversed in the anticlockwise direction.

(c) γ_2: a circle with radius $R = 1$ about $z = -1$, traversed in the clockwise direction.

(d) γ_3: a circle with radius $R = 2a$ about the origin, traversed in the anticlockwise direction.

$\big[$Check your results: (b) $I_{\gamma_1}(2) = \frac{16}{9}\pi\mathrm{i}$, (c) $I_{\gamma_2}(3) = \frac{3}{2}\pi\mathrm{i}\big]$.

Homework Problem 3: Integration by contour closure and residue theorem [5]
Points: (a)[2](M); (b)[3](A).

Calculate the following integrals (with $a, b \in \mathbb{R}$ and $a > 0$) by closing the contour with a semicircle of radius $\to \infty$ in the upper or lower complex half-planes (show that both choices give the same result!):

(a) $I(a,b) = \displaystyle\int_{-\infty}^{\infty} \mathrm{d}x \frac{x}{(x^2+b^2)(x-\mathrm{i}a)}$, (b) $I(a,b) = \displaystyle\int_{-\infty}^{\infty} \mathrm{d}x \frac{x}{(x+\mathrm{i}b)^2(x-\mathrm{i}a)}$.

[Check your results: (a) $I(3,-2) = \frac{\pi}{5}$, (b) $I(3,2) = \frac{6\pi}{25}$.]

Homework Problem 4: Inverse Fourier transform via contour closure: Green function of damped harmonic oscillator [6]
Points: (a)[3](A); (b)[3](A).

The Green function of the damped harmonic oscillator is defined by the differential equation $(\mathrm{d}_t^2 + 2\gamma \mathrm{d}_t + \Omega^2)G(t) = \delta(t)$ (with Ω and γ real and positive). Its Fourier transform, defined by $G(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega t}\tilde{\mathcal{G}}(\omega)$, is given by $\tilde{\mathcal{G}}(\omega) = (\Omega^2 - \omega^2 - 2\gamma\mathrm{i}\omega)^{-1}$. Express the Green function in the form $G(t) = \int_{-\infty}^{\infty} \mathrm{d}z\, f(z)$, and calculate the integral by closing the contour in the complex plane. You should proceed as follows:

(a) Find the residues of $f(z)$. Distinguish between the following cases:
(i) $\Omega > \gamma$ (underdamped), (ii) $\Omega = \gamma$ (critically damped) and (iii) $\Omega < \gamma$ (overdamped).
Hint: (i) and (iii) each have two poles of first order; (ii) has only a single pole, but of second order.

(b) Calculate $G(t)$ by closing the contour with an appropriately chosen semicircle with radius $R \to \infty$ (again distinguishing between the different cases!). [Check your results for $G(t)$: (i) for $\Omega = 1$ and $\gamma \to 0$, $G(\pi/2) = 1$; (ii) for $\Omega = \gamma = 1$, $G(1) = \mathrm{e}^{-1}$; (iii) for $\Omega = 4$ and $\gamma = 5$, $G(1/3) = \frac{1}{3}\mathrm{e}^{-5/3}\sinh(1)$.]

[Total Points for Homework Problems: 17]
