LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

Fakultät für Physik
R: Rechenmethoden für Physiker, WiSe 2022/23
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https://moodle.lmu.de → Kurse suchen: 'Rechenmethoden'

Sheet 14.2: Complex Analysis

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 1, 2(a,b), 4(a).
Videos exist for example problems 2 (C9.4.1), 4 (C9.4.7).

Optional Problem 1: Cauchy-Riemann equations [2]
Points: (a)[1](E); (b)[1](E).

Write the following functions of $z = x + \mathrm{i}y$ and $\bar{z} = x - \mathrm{i}y$ in the form $f(x, y) = u(x, y) + \mathrm{i}v(x, y)$ and explicitly check if the Cauchy-Riemann equations are satisfied. Which of these functions are analytic in z?

(a) $f(z) = \mathrm{e}^z$,

(b) $f(z) = \bar{z}^2$.

Optional Problem 2: Cauchy-Riemann equations [5]
Points: (a)[1](E); (b)[1](E); (c)[1](M); (d)[2](M).

Investigate, using the Cauchy-Riemann equations, which of the following functions are analytic in $z = x + \mathrm{i}y$, and if so, in which domain in $\mathbb{C}$. Check your conclusions by attempting to express each function in terms of z and $\bar{z}$.

(a) $$f(x, y) = (x^3 - 3xy^2) + \mathrm{i}(3x^2y - y^3)\,.$$

(b) $$f(x, y) = xy + \mathrm{i}\tfrac{1}{2}y^2\,.$$

(c) $$f(x, y) = \frac{x - \mathrm{i}y}{x^2 + y^2}\,.$$

(d) $$\left.\begin{matrix} f_+(x, y) \\ f_-(x, y) \end{matrix}\right\} = \mathrm{e}^x\big[x\cos y \pm y \sin y\big] + \mathrm{i}\mathrm{e}^x\big[x \sin y \mp y\cos y\big]\,.$$

Optional Problem 3: Laurent series, residues [2]
Points: (a)[0.5](E); (b)[0.5](E); (c)[1](E).

Let $p(z)$ be a polynomial of order $k \geq 0$ on $\mathbb{C}$ that does not have any zeros at z_0, then $f_m(z) = \frac{p(z)}{(z-z_0)^m}$ (with $m \geq 1$) is an analytic function on $\mathbb{C}\backslash z_0$, with a pole of order m at z_0.

(a) Show, using a Taylor series of $p(z)$ about z_0, that the Laurent series of $f_m(z)$ about z_0 has the following form:

$$f_m(z) = \sum_{n=-m}^{k-m} \frac{p^{(n+m)}(z_0)}{(n+m)!}(z - z_0)^n\,, \quad \text{with} \quad p^{(n)}(z_0) = \frac{\mathrm{d}^n}{\mathrm{d}z^n}p(z)\bigg|_{z=z_0}\,.$$

(b) For $f_m(z) = \frac{z^3}{(z-2)^m}$, find the Laurent series about the pole at $z_0 = 2$.

(c) Find the residues of $f_m(z) = \frac{z^3}{(z-2)^m}$ about the pole $z_0 = 2$ for $m = 1, 2, 3, 4$ and 5, using the formula $\mathrm{Res}(f, z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} [(z-z_0)^m f(z)]$.
[Check your results: are the residues from (c) consistent with the Laurent series of (b)?]

Optional Problem 4: Laurent series, residues [4]
Points: (a)[0.5](E); (b)[2](M); (c)[1](M); (d)[0.5](E).

For each of the following functions, determine their poles, as well as the residues using the residue formula. Then find the Laurent series about each pole using an appropriately chosen Taylor series.

(a) $\dfrac{2z^3 - 3z^2}{(z-2)^3}$, (b) $\dfrac{1}{(z-1)(z-3)}$, (c) $\dfrac{\ln z}{(z-5)^2}$, (d) $\dfrac{\mathrm{e}^{\pi z}}{(z-\mathrm{i})^m}$ with $m \geq 1$.

Hint: The Laurent series of a function of the form $f(z) = g(z)/(z-z_0)^m$, with $g(z)$ analytic in some neighbourhood of z_0, follows from the Taylor series of $g(z)$ about z_0.
[Check your results: The constant terms [coefficient of $(z-z_0)^0$] in the Laurent series are for (a) 2, (b) $-\frac{1}{4}$ for the poles at $z_0 = 1$ and 3, (c) $-\frac{1}{25}$, (d) $-\pi^m/m!$. Further check: Do the residues match the coefficients of $(z-z_0)^{-1}$ for each Laurent series?]

Optional Problem 5: Various integration contours, residue theorem [4]
Points: (a)[2](M); (a)[0.5](M); (b)[0.5](M); (c)[0.5](M); (d)[0.5](M).

Consider the function $f(z) = \dfrac{z^2}{(z^2+4)(z^2+a^2)}$, with $a \in \mathbb{R}$, $3 \leq a < 4$.

(a) Determine the residues of f at each of its poles.

Calculate the integral $I_{\gamma_i}(a) = \int_{\gamma_i} \mathrm{d}z\, f(z)$ for the following integration contours:

(b) γ_1: a circle with radius $R = 1$ about the origin, traversed in the anticlockwise direction.

(c) γ_2: a circle with radius $R = \frac{1}{2}$ about $z = 2\mathrm{i}$, traversed in the anticlockwise direction.

(d) γ_3: a circle with radius $R = 2$ about $z = 2\mathrm{i}$, traversed in the clockwise direction.

(e) γ_4: the real axis, traversed in the positive direction.

$\big[$Check your results: (c) $I_{\gamma_2}(3) = -\frac{2\pi}{5}$, (d) $I_{\gamma_3}(\frac{10}{3}) = -\frac{3\pi}{16}$, (e) $I_{\gamma_4}(\frac{7}{2}) = \frac{2\pi}{11}$.$\big]$

Optional Problem 6: Various integration contours, residue theorem [5]
Points: (a)[2.5](A); (b)[0.5](E); (c)[0.5](E); (d)[0.5](E); (e)[0.5](M); (f)[0.5](M).

Consider the function $f(z) = \dfrac{1}{[z^2 - 2az + a^2 + \frac{1}{4}]^2\,(4z^2+1)}$, with $1 < a \in \mathbb{R}$.

(a) Determine the residues of the function f at each of its poles.

Calculate the integrals $I_{\gamma_i}(a) = \int_{\gamma_i} \mathrm{d}z\, f(z)$ for the following integration contours:

(b) γ_1: a circle with radius $R = 1$ about $z_1 = 0$, traversed in the anticlockwise direction.

(c) γ_2: a circle with radius $R = \frac{1}{\sqrt{2}}a$ about $z_2 = \frac{1}{2}a(1 - \mathrm{i})$, traversed in the clockwise direction.

(d) γ_3: a circle with radius $R = a + \frac{1}{2}$ about $z_3 = \frac{1}{2}a$, traversed in the anticlockwise direction.

(e) γ_4: the line $z = x$, with $x \in (-\infty, \infty)$, traversed in the positive x-direction.

(f) γ_5: the line $z = \frac{1}{3}a + \mathrm{i}y$, with $y \in (-\infty, \infty)$, traversed in the positive y-direction.

$\big[$Check your results: (b) $I_{\gamma_1}(2) = \frac{\pi \mathrm{i}}{25}$, (c) $I_{\gamma_2}(2) = \frac{7\pi}{25}$, (e) $I_{\gamma_4}(3) = \frac{3\pi}{25}$, (f) $I_{\gamma_5}(3) = \frac{\pi \mathrm{i}}{150}$.$\big]$

[Total Points for Optional Problems: 22]
