Sheet 13: Theorems of Gauss and Stokes

Posted: Mo 23.01.23 Central Tutorial: Th 26.01.23 Due: Th 02.02.23, 14:00
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 2, 4, 6 (7, if time permits).
Videos exist for example problems 4 (V3.7.7), 7 (V3.7.11).

Example Problem 1: Gauss's theorem – cuboid (Cartesian coordinates) [2]
Points: (a)[1](M); (b)[1](M).

Consider the cuboid C, defined by $x \in (0,a)$, $y \in (0,b)$, $z \in (0,c)$, and the vector field $\mathbf{u}(\mathbf{r}) = (\frac{1}{2}x^2 + x^2 y, \frac{1}{2}x^2 y^2, 0)^T$. Compute its outward flux, $\Phi = \int_S \mathrm{d}\mathbf{S} \cdot \mathbf{u}$, through the cube's surface, $S \equiv \partial C$, in two ways:

(a) directly as a surface integral; and

(b) as a volume integral via Gauss's theorem.

[Check your results: if $a = 2$, $b = 3$, $c = \frac{1}{2}$, then $\Phi = 18$.]

Example Problem 2: Computing volume of barrel using Gauss's theorem [1]
Points: (a)[1](E); (b)[2](A,Bonus).

Consider a three-dimensional body bounded by a surface S. One method of computing its volume, V, is to express the latter as a flux integral over S by evoking Gauss's theorem for a vector field, $\mathbf{u}$, satisfying $\boldsymbol{\nabla} \cdot \mathbf{u} = 1$:

$$V = \int_V \mathrm{d}V = \int_V \mathrm{d}V \; \boldsymbol{\nabla} \cdot \mathbf{u} \overset{\text{Gauss}}{=} \int_S \mathrm{d}\mathbf{S} \cdot \mathbf{u} \, .$$

Use this method with $\mathbf{u} = \frac{1}{2}(x, y, 0)^T$ to compute, in cylindrical coordinates, the volume of

(a) a cylinder with height h and radius R, and

(b) a cylindrical barrel with height h and z-dependent radius, $\rho(z) = R[1 + a\sin(\pi z/h)]^{1/2}$, with $z \in (0,h)$ and $a > 0$. [Check your result: if $a = \pi/4$, then $V = \frac{3}{2}\pi R^2 h$.]

Example Problem 3: Gradient, divergence, curl, Laplace in cylindrical coordinates [5]
Points: (a)[0.5](E); (b)[0.5](E); (c)[0.5](E); (d)[1](M); (e)[0.5](M); (f)[1](M); (g)[1](E)

We consider a curvilinear *orthogonal* coordinate system with coordinates $\mathbf{y} = (y^1, y^2, y^3)^T \equiv (\eta, \mu, \nu)^T$, position vector $\mathbf{r}(\mathbf{y}) = \mathbf{r}(\eta, \mu, \nu)$ and coordinate basis vectors $\partial_\eta \mathbf{r} = \mathbf{e}_\eta n_\eta$, $\partial_\mu \mathbf{r} = \mathbf{e}_\mu n_\mu$, $\partial_\nu \mathbf{r} = \mathbf{e}_\nu n_\nu$, with $\|\mathbf{e}_j\| = 1$ and norm factors n_η, n_μ, n_ν (i.e. no summations over η, μ and ν here!). Furthermore, let $f(\mathbf{r})$ be a scalar field and $\mathbf{u}(\mathbf{r}) = \mathbf{e}_\eta u^\eta + \mathbf{e}_\mu u^\mu + \mathbf{e}_\nu u^\nu$ a vector field, expressed in the *local basis*. Then, the gradient, divergence, curl and Laplace operator are given by

$$\nabla f = \mathbf{e}_\eta \frac{1}{n_\eta}\partial_\eta f \; + \; \circlearrowright^{\eta\,\mu}_{\nu} \; + \; \circlearrowright^{\eta\,\mu}_{\nu} \,,$$

$$\nabla \cdot \mathbf{u} = \frac{1}{n_\eta n_\mu n_\nu}\partial_\eta\left(n_\mu n_\nu u^\eta\right) \; + \; \circlearrowright^{\eta\,\mu}_{\nu} \; + \; \circlearrowright^{\eta\,\mu}_{\nu} \,,$$

$$\nabla \times \mathbf{u} = \mathbf{e}_\eta \frac{1}{n_\mu n_\nu}\Big[\partial_\mu\left(n_\nu u^\nu\right) - \partial_\nu\left(n_\mu u^\mu\right)\Big] \; + \; \circlearrowright^{\eta\,\mu}_{\nu} \; + \; \circlearrowright^{\eta\,\mu}_{\nu} \,,$$

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{n_\eta n_\mu n_\nu}\partial_\eta\left(\frac{n_\mu n_\nu}{n_\eta}\partial_\eta f\right) \; + \; \circlearrowright^{\eta\,\mu}_{\nu} \; + \; \circlearrowright^{\eta\,\mu}_{\nu} \,,$$

where circles with three arrows denote cyclical permutations of indices. Now consider the cylindrical coordinates defined by $\mathbf{r}(\rho, \phi, z) = (\rho\cos\phi, \rho\sin\phi, z)^T$.

(a) Write down formulas for $\mathbf{e}_\rho$, $\mathbf{e}_\phi$, $\mathbf{e}_z$ and n_ρ, n_ϕ, n_z.

Starting from the general formulas given above, find explicit formulas for

(b) ∇f, (c) $\nabla \cdot \mathbf{u}$, (d) $\nabla \times \mathbf{u}$, (e) $\nabla^2 f$.

(f) Verify explicitly that $\nabla \times (\nabla f) = \mathbf{0}$, using the given formulae for the gradient and curl in general curvilinear coordinates η, μ, ν (i.e. not specifically cylindrical coordinates).

(g) Use cylindrical coordinates to compute ∇f, $\nabla \cdot \mathbf{u}$, $\nabla \times \mathbf{u}$ and $\nabla^2 f$ for the fields $f(\mathbf{r}) = \|\mathbf{r}\|^2$ and $\mathbf{u}(\mathbf{r}) = (x, y, 2z)^T$. [Check your results: if $\mathbf{r} = (1, 1, 1)^T$, then $\nabla f = (2, 2, 2)^T$, $\nabla \cdot \mathbf{u} = 4$, $\nabla \times \mathbf{u} = \mathbf{0}$ and $\nabla^2 f = 6$.]

Example Problem 4: Gradient, divergence, curl (spherical coordinates) [2]

Consider the scalar field $f(\mathbf{r}) = \frac{1}{r}$ and the vector field $\mathbf{u}(\mathbf{r}) = (\mathrm{e}^{-r/a}/r)\mathbf{r}$, with $\mathbf{r} = (x, y, z)^T$ and $r = \sqrt{x^2 + y^2 + z^2}$. Calculate ∇f, $\nabla \cdot \mathbf{u}$, $\nabla \times \mathbf{u}$ and $\nabla^2 f$ explicitly for $r > 0$,

(a) in Cartesian coordinates; (b) in spherical coordinates.

Verify that your results from (a) and (b) are consistent with one another.

Example Problem 5: Gauss's theorem – cylinder (cylindrical coordinates) [2]
Points: (a)[0.5](E); (b)[1](M); (c)[0.5](M)

Consider a vector field, $\mathbf{u}$, defined in cylindrical coordinates by $\mathbf{u}(\mathbf{r}) = \mathbf{e}_\rho z\rho$, and a cylindrical volume, V, defined by $\rho \in (0, R)$, $\phi \in (0, 2\pi)$, $z \in (0, H)$.

(a) Compute the divergence of the vector field $\mathbf{u}$ in cylindrical coordinates.

Compute the flux, Φ, of the vector field $\mathbf{u}$ through the surface, S, of the cylindrical volume V, via two methods:

(b) by calculating the surface integral, $\Phi = \int_S \mathrm{d}\mathbf{S} \cdot \mathbf{u}$, explicitly;

(c) by using Gauss's theorem to convert the flux integral to a volume integral of $\nabla \cdot \mathbf{u}$ and then computing the volume integral explicitly.

Example Problem 6: Stokes's theorem – magnetic dipole (spherical coordinates) [2]
Points: (a)[1](M); (b)[1](M)

Every magnetic field can be represented as $\mathbf{B} = \nabla \times \mathbf{A}$, where the vector field $\mathbf{A}$ is known as the **vector potential** of the field. For a magnetic dipole,

$$\mathbf{A} = \frac{1}{c}\frac{\mathbf{m} \times \mathbf{r}}{r^3}, \qquad \mathbf{B} = \frac{1}{c}\frac{3\mathbf{r}(\mathbf{m}\cdot\mathbf{r}) - \mathbf{m}r^2}{r^5},$$

where c is the speed of light. Let the constant dipole moment $\mathbf{m}$ be oriented in the z-direction, $\mathbf{m} = \mathbf{e}_z m$. Let H be a hemisphere with radius R, oriented with base surface in the xy-plane, symmetry axis along the positive z-axis and 'north pole' on the latter. Compute the flux integral of the magnetic field through this hemisphere, $\Phi_H = \int_H \mathrm{d}\mathbf{S} \cdot \mathbf{B}$, in two different ways:

(a) directly, using spherical coordinates;

(b) use $\mathbf{B} = \nabla \times \mathbf{A}$ and Stokes's theorem to express Φ as a line integral of $\mathbf{A}$ over the boundary of the surface of H, and evaluate the line integral.

Example Problem 7: Stokes's theorem – magnetic field of a current carrying conductor (cylindrical coordinates) [4]
Points: (a)[1](E); (b)[1](M); (c)[0.5](M); (d)[0.5](E); (e)[0.5](M); (f)[0.5](M)

Let an infinitely long, infinitesimally thin conductor be oriented along the z-axis and carry a current I. It generates a magnetic field of the following form:

$$\mathbf{B}(\mathbf{r}) = \frac{2I}{c}\frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \mathbf{e}_\phi \frac{2I}{c}\frac{1}{\rho}, \qquad \text{for} \quad \rho = \sqrt{x^2+y^2} > 0.$$

Calculate the divergence and rotation of $\mathbf{B}(\mathbf{r})$ explicitly for $\rho > 0$, using

(a) Cartesian coordinates; and

(b) cylindrical coordinates. [Compare your results from (a) and (b)!]

(c) Use cylindrical coordinates to compute the line integral, $\oint_\gamma \mathrm{d}\mathbf{r} \cdot \mathbf{B}$, of the magnetic field along the edge, γ, of a circular disk, D, with radius $R > 0$, centred on the z-axis, and oriented parallel to the xy-plane.

(d) Use Stokes's theorem and the result from (c) to compute the flux integral, $\int_D \mathrm{d}\mathbf{S} \cdot (\nabla \times \mathbf{B})$, of the curl of the magnetic field over the disk D prescribed in (c).

(e) Use your results for $\nabla \times \mathbf{B}$ from (a) and (d) to argue that the curl of the field is proportional to a two-dimensional δ-function, $\nabla \times \mathbf{B} = \mathbf{e}_z\, C\delta(x)\delta(y)$. Find the constant C. [*Hint*: The two-dimensional δ-function is normalized such that $\int_D \mathrm{d}S\, \delta(x)\delta(y) = 1$ for the area integral over any surface D which lies parallel to the xy-plane and intersects the z-axis.]

(f) Write the result obtained in (e) in the form $\boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j}(\mathbf{r})$ and determine $\mathbf{j}(\mathbf{r})$. This equation is Ampere's law (one of the Maxwell equations), where $\mathbf{j}(\mathbf{r})$ is the current density. Can you give a physical interpretation of your result for $\mathbf{j}(\mathbf{r})$?

[Total Points for Example Problems: 18]

Homework Problem 1: Stokes's theorem – cuboid (Cartesian coordinates) [2]
Points: (a)[1](M); (b)[1](M).

Consider the cuboid C, defined by $x \in (0,a)$, $y \in (0,b)$, $z \in (0,c)$, and the vector field $\mathbf{w}(\mathbf{r}) = \frac{1}{2}(yz^2, -xz^2, 0)^T$. Compute the outward flux of its curl, $\Phi = \int_S \mathrm{d}\mathbf{S}\cdot(\boldsymbol{\nabla}\times\mathbf{w})$, through the surface $S \equiv \partial C\backslash\text{top}$, consisting of all faces of the cube except the top one at $z = c$, in two ways:

(a) directly as a surface integral;

(b) as a line integral via Stokes's theorem.

[Check your results: if $a = 2$, $b = 3$, $c = \frac{1}{2}$, then $\Phi = \frac{3}{2}$.]

Homework Problem 2: Computing volume of grooved ball using Gauss's theorem [1]
Points: (a)[1](E); (b)[2](A,Bonus).

The volume of a body can be computed using a surface integral, $V = \int_S \mathrm{d}\mathbf{S}\cdot\frac{1}{3}\mathbf{r}$, over the body's surface, S (cf. the corresponding example problem). Use this method to compute, in spherical coordinates,

(a) the volume, V, of a ball with radius R, and

(b) the volume, $V(\epsilon, n)$, of a 'grooved ball', whose ϕ-dependent radius is described by the function $r(\phi) = R\big[1 + \epsilon\sin\big(n\phi\big)\big]^{2/3}$, where $1 \leq n \in \mathbb{N}$ determines the number of grooves and $\epsilon < 1$ their depth. [Check your result: $V(\frac{1}{4}, 4) = \frac{33}{32}V(0,0)$.]

Homework Problem 3: Gradient, divergence, curl, Laplace in spherical coordinates [5]
Points: (a)[0.5](E); (b)[0.5](E); (c)[0.5](E); (d)[1](M); (e)[0.5](M); (f)[1](M); (g)[1](E)

Consider a curvilinear *orthogonal* coordinate system with coordinates $\mathbf{y} = (y^1, y^2, y^3)^T \equiv (\eta, \mu, \nu)^T$, position vector $\mathbf{r}(\mathbf{y}) = \mathbf{r}(\eta,\mu,\nu)$ and coordinate basis vectors $\partial_\eta\mathbf{r} = \mathbf{e}_\eta n_\eta$, $\partial_\mu\mathbf{r} = \mathbf{e}_\mu n_\mu$, $\partial_\nu\mathbf{r} = \mathbf{e}_\nu n_\nu$, with $\|\mathbf{e}_j\| = 1$. Furthermore, $f(\mathbf{r})$ is a scalar field and $\mathbf{u}(\mathbf{r}) = \mathbf{e}_\eta u^\eta + \mathbf{e}_\mu u^\mu + \mathbf{e}_\nu u^\nu$ is a vector field, expressed in the *local basis*. Then, the gradient, divergence, curl and Laplace operator are given by

$$\boldsymbol{\nabla} f = \mathbf{e}_\eta\frac{1}{n_\eta}\partial_\eta f \; + \overset{\eta\;\;\mu}{\underset{\nu}{\circlearrowright}} + \overset{\eta\;\;\mu}{\underset{\nu}{\circlearrowright}} \,,$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{1}{n_\eta n_\mu n_\nu}\partial_\eta\left(n_\mu n_\nu u^\eta\right) \; + \overset{\eta\;\;\mu}{\underset{\nu}{\circlearrowright}} + \overset{\eta\;\;\mu}{\underset{\nu}{\circlearrowright}} \,,$$

$$\boldsymbol{\nabla}\times\mathbf{u} = \mathbf{e}_\eta\frac{1}{n_\mu n_\nu}\Big[\partial_\mu\left(n_\nu u^\nu\right) - \partial_\nu\left(n_\mu u^\mu\right)\Big] \; + \overset{\eta\;\;\mu}{\underset{\nu}{\circlearrowright}} + \overset{\eta\;\;\mu}{\underset{\nu}{\circlearrowright}} \,,$$

$$\boldsymbol{\nabla}^2 f = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}f) = \frac{1}{n_\eta n_\mu n_\nu}\partial_\eta\left(\frac{n_\mu n_\nu}{n_\eta}\partial_\eta f\right) \; + \overset{\eta\;\;\mu}{\underset{\nu}{\circlearrowright}} + \overset{\eta\;\;\mu}{\underset{\nu}{\circlearrowright}} \,.$$

Consider the spherical coordinates defined by $\mathbf{r}(r,\theta,\phi) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)^T$.

(a) Write down formulas for $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ and n_r, n_θ, n_ϕ.

Starting from the general formulas given above, find an explicit formula for

(b) $\boldsymbol{\nabla} f$, (c) $\boldsymbol{\nabla}\cdot\mathbf{u}$, (d) $\boldsymbol{\nabla}\times\mathbf{u}$, (e) $\boldsymbol{\nabla}^2 f$.

(f) Verify explicitly that $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{u}) = 0$, using the above formulae for the divergence and the curl for general curvilinear coordinates η, μ, ν (i.e. not specifically spherical coordinates).

(g) Use spherical coordinates to compute $\boldsymbol{\nabla} f$, $\boldsymbol{\nabla}\cdot\mathbf{u}$, $\boldsymbol{\nabla}\times\mathbf{u}$ and $\boldsymbol{\nabla}^2 f$ for the fields $f(\mathbf{r}) = \|\mathbf{r}\|^2$ and $\mathbf{u}(\mathbf{r}) = (0,0,z)^T$. [Check your results: if $\mathbf{r} = (1,1,1)^T$, then $\boldsymbol{\nabla} f = (2,2,2)^T$, $\boldsymbol{\nabla}\cdot\mathbf{u} = 1$, $\boldsymbol{\nabla}\times\mathbf{u} = \mathbf{0}$ and $\boldsymbol{\nabla}^2 f = 6$.]

Homework Problem 4: Gradient, divergence, curl (cylindrical coordinates) [2]
Points: (a)[1](E); (b)[1](M)

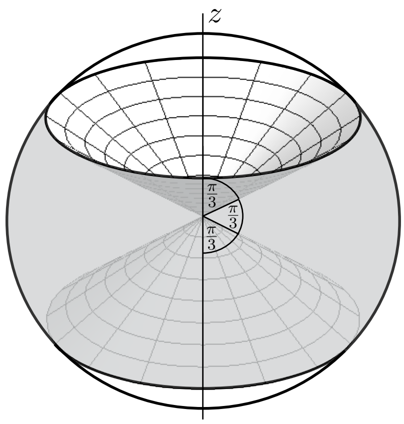
Consider the scalar field $f(\mathbf{r}) = z(x^2+y^2)$ and the vector field $\mathbf{u}(\mathbf{r}) = (zx, zy, 0)^T$. Calculate $\boldsymbol{\nabla} f$, $\boldsymbol{\nabla}\cdot\mathbf{u}$, $\boldsymbol{\nabla}\times\mathbf{u}$ and $\boldsymbol{\nabla}^2 f$ explicitly in

(a) Cartesian coordinates; (b) cylindrical coordinates.

Verify that your results from (a) and (b) are consistent with one another.

Homework Problem 5: Gauss's theorem – wedge ring (spherical coordinates) [4]
Points: (a)[1](M); (b)[2](A); (c)[1](M)

Consider the 'wedge–ring', W, which is shaded grey in the sketch. This shape can be expressed in spherical coordinates by the conditions $r \in (0,R)$ and $\theta \in (\pi/3, 2\pi/3)$. (Such a ring-like object, with wedge-shaped inner profile and rounded outer profile, is constructed from a sphere with radius R, by removing a double cone centred on the z-axis with apex angle $\pi/3$.) Compute the outward flux, Φ_W, of the vector field $\mathbf{u}(\mathbf{r}) = \mathbf{e}_r r^2$ through the surface, ∂W, of the wedge–ring, in two different ways:



(a) Compute the flux integral, $\Phi_W = \int_{\partial W} \mathrm{d}\mathbf{S}\cdot\mathbf{u}$. [Check your result: if $R = \frac{1}{2}$, then $\Phi_W = \frac{\pi}{8}$.]

(b) Use Gauss's theorem to convert the flux integral into a volume integral of the divergence $\boldsymbol{\nabla}\cdot\mathbf{u}$, and compute the volume integral explicitly. *Hint:* In the local basis of spherical coordinates,

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{1}{r^2}\partial_r\left(r^2 u^r\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta\, u^\theta\right) + \frac{1}{r\sin\theta}\partial_\phi u^\phi\ .$$

(c) For the vector field $\mathbf{w}(\mathbf{r}) = -\mathbf{e}_\theta\cos\theta$, calculate the outward flux, $\tilde{\Phi}_W = \int_{\partial W}\mathrm{d}\mathbf{S}\cdot\mathbf{w}$, through the surface of the wedge–ring, either directly or by using Gauss's theorem.
[Check your result: if $R = \frac{1}{\sqrt{3}}$, then $\tilde{\Phi}_W = \frac{\pi}{\sqrt{12}}$.]

Homework Problem 6: Stokes's theorem – cylinder (cylindrical coordinates) [2]
Points: (a)[1](E); (b)[1](E)

Consider a cylinder, C, with radius R and height aR^2, centred on the z-axis, with base in the xy-plane, and the vector field $\mathbf{u} = \frac{x^2+y^2}{z}(-y, x, 0)^T$. Compute the flux of its curl, $\Phi_T = \int_T \mathrm{d}\mathbf{S}\cdot(\boldsymbol{\nabla}\times\mathbf{u})$, through the top face, T, of the cylinder in two different ways:

(a) directly, using cylindrical coordinates; and

(b) by using Stokes's theorem to express Φ_T as a line integral of $\mathbf{u}$ over the boundary, ∂T, of the cylinder top, and then computing the integral.

Homework Problem 7: Gauss's law – electric field of a point charge (spherical coordinates) [4]
Points: (a)[1](E); (b)[1](M); (c)[0.5](M); (d)[0.5](E); (e)[0.5](M); (f)[0.5](M)

The electric field of a point charge Q at the origin has the form

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{r^3}\mathbf{r} = \mathbf{e}_r\frac{Q}{r^2}\,, \qquad \text{with} \quad r > 0, \qquad r = \sqrt{x^2+y^2+z^2}\,.$$

Calculate the divergence and the curl of $\mathbf{E}(\mathbf{r})$ explicitly for $r > 0$, using

(a) Cartesian coordinates; and

(b) spherical coordinates. [Compare your results from (a) and (b)!]

(c) Use spherical coordinates to compute the flux, $\Phi_S = \int_S \mathrm{d}\mathbf{S}\cdot\mathbf{E}$, of the electric field through a sphere, S, with radius $R > 0$, centered at the origin.

(d) Use Gauss's theorem and the result from (c) to compute the integral, $\int_V \mathrm{d}V(\boldsymbol{\nabla}\cdot\mathbf{E})$, over the volume, V, enclosed by the sphere S described in (c).

(e) Use your results for $\boldsymbol{\nabla}\cdot\mathbf{E}$ from (a) and (d) to argue that the divergence of the field is proportional to a three-dimensional δ-function, i.e. has the form $\boldsymbol{\nabla}\cdot\mathbf{E} = C\,\delta^{(3)}(\mathbf{r})$. Find the constant C. [Hint: The normalization of $\delta^{(3)}(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$ is given by the volume integral $\int_V \mathrm{d}V\,\delta^{(3)}(\mathbf{r}) = 1$, for any volume, V, that contains the origin.]

(f) Write your result from (e) in the form $\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho(\mathbf{r})$, and determine $\rho(\mathbf{r})$. This equation is the (physical) Gauss's law (one of the Maxwell equations), where $\rho(\mathbf{r})$ is the charge density. Can you interpret your result in terms of $\rho(\mathbf{r})$?

[Total Points for Homework Problems: 20]
