Sheet 13: Theorems of Gauss and Stokes

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 2, 4, 6 (7, if time permits).
Videos exist for example problems 4 (V3.7.7), 7 (V3.7.11).

Optional Problem 1: Gauss's theorem – cube (Cartesian coordinates) [2]
Points: (a)[1](M); (b)[1](M).

Consider the cube C, defined by $x \in (0, a)$, $y \in (0, a)$, $z \in (0, a)$, and the vector field $\mathbf{u}(\mathbf{r}) = (x^2, y^2, z^2)^T$. Compute its outward flux, $\Phi = \int_S \mathrm{d}\mathbf{S} \cdot \mathbf{u}$, through the cube's surface, $S \equiv \partial C$, in two ways:

(a) directly as a surface integral;

(b) as a volume integral via Gauss's theorem.

[Check your result: if $a = 2$, then $\Phi = 48$.]

Optional Problem 2: Stokes's theorem – cube (Cartesian coordinates) [2]
Points: (a)[1](M); (b)[1](M).

Consider the cube C, defined by $x \in (0, a)$, $y \in (0, a)$, $z \in (0, a)$, and the vector field $\mathbf{w}(\mathbf{r}) = (-y^2, x^2, 0)^T$. Compute the outward flux of its curl, $\Phi = \int_S \mathrm{d}\mathbf{S} \cdot (\boldsymbol{\nabla} \times \mathbf{w})$, through the surface $S \equiv \partial C \backslash$top, consisting of all faces of the cube except the top one at $z = a$, in two ways:

(a) directly as a surface integral;

(b) as a line integral via Stokes's theorem.

[Check your result: if $a = 2$, then $\Phi = -16$.]

Optional Problem 3: Gauss's theorem – electrical dipole potential (spherical coordinates) [4]
Points: (a)[0.5](E); (b)[0.5](E); (c)[1](M); (d)[1](M); (e)[0.5](M); (f)[0.5](M).

The potential of an electric dipole with dipole moment $\mathbf{p} = p\mathbf{e}_z$ is given by

$$\Phi(\mathbf{r}) = \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} = \frac{pz}{r^3}$$

(a) Calculate the electric field, $\mathbf{E} = -\boldsymbol{\nabla}\Phi(\mathbf{r})$, explicitly in Cartesian coordinates.

(b) Represent $\Phi(\mathbf{r})$ in spherical coordinates and calculate the electric field explicitly in spherical coordinates. Compare this result with the result obtained in (a).
Hint: $\mathbf{e}_z = \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta$.

(c) Calculate the divergence and the curl of the electric field explicitly in Cartesian coordinates.

(d) Calculate the divergence and the curl of the electric field explicitly in spherical coordinates. [Compare the results obtained in (b) and (c)!]

(e) According to the (physical) law of Gauss, we have $\int_S \mathrm{d}\mathbf{S} \cdot \mathbf{E} = 4\pi Q$, where Q is the total charge contained within the volume of S. Now consider a sphere, S, of radius R, centered at the origin. Calculate Q by performing the flux integral over the sphere. Does your result for Q make physical sense? Explain!

(f) Now compute the flux integral from (e) in an alternative manner: convert it via the (mathematical) theorem of Gauss into a volume integral over $\boldsymbol{\nabla} \cdot \mathbf{E}$, and evaluate this integral using the result from (d). Comment on the behavior of the integrand at $r = 0$.

[Total Points for Optional Problems: 8]
