Sheet 12.2: Fourier Integrals, Differential Equations

Posted: Mo 16.01.23 Central Tutorial: Fr(!) 20.01.23 Due: Th 26.01.23, 14:00
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 3, 4, 5.
Videos exist for example problems 2 (C6.3.3), 3 (C7.5.1).

Example Problem 1: Properties of the Fourier Transform [1]
Points: (a)[0.5](E); (b)[0.5](E).

Demonstrate the following properties of the Fourier transformation, where a is an arbitrary real constant.

(a) The Fourier transform of $f(x-a)$ is $e^{-ika}\tilde{f}(k)$.

(b) The Fourier transform of $f(ax)$ is $\tilde{f}(k/a)/|a|$, where $a \neq 0$.

Example Problem 2: Fourier transform of a Gaussian function [2]
Points: [2](E).

Show that the Fourier transform of a normalized Gaussian distribution with width σ, $g^{[\sigma]}(x) = \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-x^2/2\sigma^2}$, with $\int_{-\infty}^{\infty} \mathrm{d}x\, g^{[\sigma]}(x) = 1$, is given by $\tilde{g}_k^{[\sigma]} = \mathrm{e}^{-\sigma^2 k^2/2}$. *Hint*: The Fourier integral can be calculated by completing the square in the exponent.

Example Problem 3: Green function of $(\mathrm{d}_t + a)$ [4]
Points: (a)[1](M); (b)[0.5](E); (c)[0.5](E); (d)[1](E); (e)[1](M).

Let $\hat{L}(t) = (\mathrm{d}_t + a)$ be a first-order differential operator, and a be a positive, real constant. The corresponding Green function is defined by the differential equation:

$$\hat{L}(t)G(t) = \delta(t)\ . \tag{1}$$

(a) Show that the ansatz

$$G(t) = \Theta(t)x_h(t) \qquad \text{with } \Theta(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases},$$

satisfies the defining equation (1), provided that $x_h(t)$ is a solution to the homogeneous equation $\hat{L}(t)x_h(t) = 0$ with initial condition $x_h(0) = 1$. [Hint: The initial condition guarantees that $\delta(t)x_h(t) = \delta(t)$.]

(b) Determine $G(t)$ explicitly by solving the homogeneous equation for $x_h(t)$. [Check your result: $G(\frac{1}{a}\ln 2) = \frac{1}{2}$.]

(c) Calculate the Fourier integral $\tilde{\mathcal{G}}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} G(t)$. [Check your result: for $a = 1$, $|\tilde{\mathcal{G}}(a)| = \frac{1}{\sqrt{2}}$.]

(d) Consistency check: Alternatively, determine $\tilde{\mathcal{G}}(\omega)$ via a Fourier transformation of the defining equation (1). Is the result in agreement with the result from part (c) of the exercise?

(e) Find a solution to the inhomogeneous differential equation, $(\mathrm{d}_t + a)x(t) = \mathrm{e}^{2at}$, by convolving the function $G(t)$ with the inhomogeneity. Verify the obtained solution explicitly by inserting it into the differential equation.

Example Problem 4: Fixed point of a differential equation in one dimension [2]
Points: (a)[0.5]; (b)[0.5]; [1](M).

Consider the autonomous differential equation $\dot{x} = f_\lambda(x) = (x^2 - \lambda)^2 - \lambda^2$ for the real function $x(t)$, with $\lambda \in \mathbb{R}$.

(a) Find the fixed points of this differential equation as a function of λ for (i) $\lambda \leq 0$, and (ii) $\lambda > 0$. [Check your results: for $\lambda = 2$, the fixed points lie at 0, 2, and -2.]

(b) Make two separate sketches of $f(x)$ as a function of x for the following fixed values of λ: (i) $\lambda = -1$ and (ii) $\lambda = +1$, and mark on your sketches the fixed points found in (a).

(c) Determine the stability of each of these fixed points via a graphical analysis of the function, and show the flow of $x(t)$ in the neighbourhood of these fixed points on the sketch from (b).

Example Problem 5: Stability analysis in two dimensions [6]
Points: (a)[0.5](E); (b)[0.5](E); (c)[1](E); (d)[2](E); (e)[2](M).

The function $\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto \mathbf{x}(t)$ satisfies the following differential equation, with $0 < c \in \mathbb{R}$:

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} 2x^2 - xy \\ c(1-x) \end{pmatrix}.$$

(a) Find the fixed point, $\mathbf{x}^*$, of the differential equation.

(b) For a small displacement, $\boldsymbol{\eta} = \mathbf{x} - \mathbf{x}^*$, from the fixed point, linearize the differential equation and bring it into the form $\dot{\boldsymbol{\eta}} = A\boldsymbol{\eta}$. Find the matrix A.

(c) Check that the matrix elements of A are given by $A^i_{\ j} = \left(\frac{\partial f^i}{\partial x^j}\right)|_{\mathbf{x}=\mathbf{x}^*}$.

(d) Find the eigenvalues and eigenvectors of A.

(e) Analyze the stability of the fixed point: For displacements relative to the fixed point, in which directions do these displacements grow or shrink the fastest? On which timescales?

[Check your results: if $c = 3$, then (a) $\|\mathbf{x}^*\| = \sqrt{5}$, (b) $\det A = -3$, (d) eigenvalues: $\lambda_+ = 3$, $\lambda_- = -1$; eigenvectors: $\mathbf{v}_+ = (3, -3)^T$ and $\mathbf{v}_- = (1, 3)^T$.]

Example Problem 6: Sketching field lines in two dimensions [2]
Points: [2](E).

The behavior of a vector field, $\mathbf{u}(\mathbf{r})$, is often indicated graphically by sketching its field lines. A field line is a curve such that the tangent vector at any point along the curve points in the direction of the field at that point. If $\mathbf{r}(t)$ is a parametrization of a field line, its shape is thus determined by the requirement $\dot{\mathbf{r}}(t)||\mathbf{u}(\mathbf{r}(t))$. This can be used to set up a differential equation whose solution describes the shape of the field lines.
To illustrate the procedure, let us consider a two-dimensional vector field in two dimensions, $\mathbf{u} : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{r} = (x, y)^T \mapsto \mathbf{u}(\mathbf{r}) = (u_x(\mathbf{r}), u_y(\mathbf{r}))^T$. Let $\mathbf{r}(t) = (x(t), y(t))^T$ be a parametrization of a field line, then the components of its tangent vector satisfy the equation

$$\frac{\dot{y}(t)}{\dot{x}(t)} = \frac{u_y(\mathbf{r}(t))}{u_x(\mathbf{r}(t))}.$$

Alternatively, we can parametrize the field line as $(x, y(x))^T$, viewing y as function of x. If $x(t)$ changes as a funtion of time, so does $y(t) = y(x(t))$, in a manner satisfying the relation $\dot{y}(t) = \frac{\mathrm{d}y(x(t))}{\mathrm{d}x}\dot{x}(t)$, or $\frac{\mathrm{d}y(x(t))}{\mathrm{d}x} = \frac{\dot{y}(t)}{\dot{x}(t)}$. Combining this with the above equation, we obtain

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = \frac{u_y(\mathbf{r})}{u_x(\mathbf{r})}.$$

This is a differential equation for $y(x)$, whose solution describes the shape of the field lines. Different choices for the initial conditions of the DEQ yield different field lines.

Consider the vector field $\mathbf{u}(\mathbf{r}) = (-ay, x)^T$, with $a > 0$. Set up and solve a differential equation for its field lines, $y(x)$. Sketch some representative lines for the case $a = \frac{1}{2}$. [Check your results: for $a = \frac{1}{2}$, a field line passing through the point $(x, y)^T = (3, 0)^T$ also passes through $(1, 4)^T$.]

[Total Points for Example Problems: 17]

Homework Problem 1: Properties of the Fourier transform [2]
Points: (a)[0.5](E); (b)[0.5](E); (c)[1](M).

Prove that the following properties of the Fourier transform hold in 2 dimensions, where $\mathbf{a} \in \mathbb{R}^2$, $\alpha \in \mathbb{R} \setminus \{0\}$ and R is a rotation matrix.

(a) The Fourier transform of $f(\mathbf{x} - \mathbf{a})$ is $\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{a}}\tilde{f}(\mathbf{k})$.

(b) The Fourier transform of $f(\alpha\mathbf{x})$ is $\frac{1}{|\alpha|^2}\tilde{f}(\mathbf{k}/\alpha)$.

(c) The Fourier transform of $f(R\mathbf{x})$ is $\tilde{f}(R\mathbf{k})$.

Homework Problem 2: Convolution of Gaussian Functions [6]
Points: (a)[1](M); (b)[1](E); (c)[1](M); (d)[1](M); (e)[1](M); (f)[1](E)

The purpose of this exercise is to illustrate the following statement: 'The fine structure of a function (e.g. noise in a test signal) can be smoothed out via convolution with a peaked function of suitable width.'

A normalized Gaussian function with width σ has the form $g^{[\sigma]}(x) = \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-x^2/2\sigma^2}$. Show that the convolution of two normalized Gaussians with widths σ_1 and σ_2 is again a normalized Gaussian, with width $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$, i.e. show that $\left(g^{[\sigma_1]} * g^{[\sigma_2]}\right)(x) = g^{[\sigma]}(x)$. Do this via two different methods, (a) and (b):

(a) Calculate the convolution integral by completing the square in the exponent.

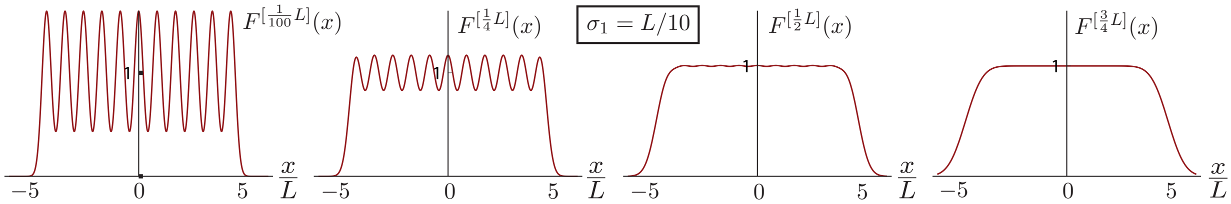
(b) Use the convolution theorem, $\big(\widetilde{g^{[\sigma_1]} * g^{[\sigma_2]}}\big)(k) = \tilde{g}^{[\sigma_1]}(k)\tilde{g}^{[\sigma_2]}(k)$, and the known form of the Fourier transform (from the example problem) of a Gaussian, $\tilde{g}^{[\sigma_j]}(k)$.

(c) Draw two qualitative sketches, the first of $g^{[\sigma_1]}(x)$, $g^{[\sigma_2]}(x)$ and $g^{[\sigma]}(x)$, the second of their respective Fourier spectra $\tilde{g}^{[\sigma_1]}(k)$, $\tilde{g}^{[\sigma_2]}(k)$ and $\big(\widetilde{g^{[\sigma_1]} * g^{[\sigma_2]}}\big)(k)$. Explain, with reference to the sketch, why the convolution of a function (here $g^{[\sigma_1]}$) with a peaked function (here $g^{[\sigma_2]}$) leads to a broadened version of the first function.

Let $f^{[\sigma_1]}(x) = \sum_{n=-5}^{5} g_n^{[\sigma_1]}(x)$, with $g_n^{[\sigma_1]}(x) = g^{[\sigma_1]}(x - nL)$, be a 'comb' of 11 identical, normalized Gaussian peaks of width σ_1, with peak-to-peak distance L, and let $F^{[\sigma_2]}(x) = \big(f * g^{[\sigma_2]}\big)(x)$ be the convolution of this comb with a normalized Gaussian peak of width σ_2.

(d) Find a formula for $F^{[\sigma_2]}(x)$, expressed as a sum over the normalized Gaussians. What is the width of each of these peaks?

(e) The sketch shows $F^{[\sigma_2]}(x)$ for $\sigma_1/L = \frac{1}{4}$ and four values of σ_2/L: $\frac{1}{100}$, $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$. Explain the observed behaviour based on your formula from part (d) of the exercise. Why does the fine structure vanish in $F^{[\sigma_2]}(x)$ for $\sigma_2 \gtrsim \frac{1}{2}L$?



(f) Regarding the introductory statement about using convolutions to smoothen noisy functions: explain in general how the width of the peaked function should be chosen to smooth out the noise.

Homework Problem 3: Green function of critically damped harmonic oscillator [4]
Points: (a)[1](M); (b)[0.5](E); (c)[0.5](E); (d)[1](E); (e)[1](M).

A driven, critically damped harmonic oscillator with frequency $\Omega > 0$ and damping rate $\gamma = \Omega$ satisfies the equation $\hat{L}(t)\, q(t) = g(t)$, with $\hat{L}(t) = (\mathrm{d}_t^2 + 2\Omega \mathrm{d}_t + \Omega^2)$. The corresponding Green function is defined by the differential equation

$$\hat{L}(t)\, G(t) = \delta(t) \; . \tag{2}$$

(a) Show that the ansatz

$$G(t) = \Theta(t) q_h(t) \, , \qquad \text{with} \qquad \Theta(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} ,$$

satisfies the defining equation (2) if $q_h(t)$ is a solution of the homogeneous equation $\hat{L}(t)\, q_h(t) = 0$, with initial values $q_h(0) = 0$ and $\mathrm{d}_t q_h(0) = 1$. [Hint: the initial values ensure that $\delta(t) q_h(t) = \delta(t) q_h(0) = 0$ and $\delta(t)\mathrm{d}_t q_h(t) = \delta(t)\mathrm{d}_t q_h(0) = \delta(t)$.]

(b) Determine $G(t)$ explicitly by solving the homogeneous equation for $q_h(t)$, using the ansatz $q_h(t) = (c_1 + c_2 t)\mathrm{e}^{-\Omega t}$ (cf. homework problem 4(b) from sheet 10). [Check your result: if $\Omega = 1$, then $G(1) = 1/\mathrm{e}$.]

(c) Compute the Fourier integral $\tilde{\mathcal{G}}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} G(t)$. [Check your result: if $\Omega = 1$, then $|\tilde{\mathcal{G}}(\Omega)| = \frac{1}{2}$.]

(d) Consistency check: find $\tilde{\mathcal{G}}(\omega)$ in an alternative way by Fourier transforming the defining equation (2). Does the result agree with that of subproblem (c)?

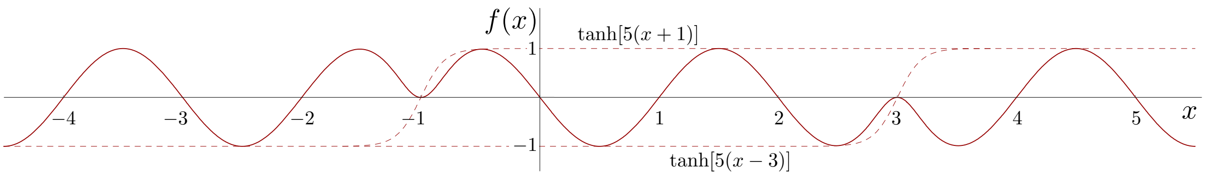
(e) Find a solution of the inhomogeneous differential equation, $\hat{L}(t)\, q(t) = g_0 \sin(\omega_0 t)$, by convolving $G(t)$ with the inhomogeneity. Check explicitly that your result satisfies this equation. [Hint: it is advisable to represent the sine function by $\mathrm{Im}\left[\mathrm{e}^{\mathrm{i}\omega_0 t}\right]$ and use $\mathrm{e}^{\mathrm{i}\omega_0 t}$ as inhomogeneity, and to take the imaginary part only at the very end of the calculation.]

Homework Problem 4: Fixed points of a differential equation in one dimension [2]
Points: (a)[0.5](E); (b)[0.5](E); (c)[1](M).

Consider the differential equation $\dot{x} = f(x) = \tanh[5(x-3)]\tanh[5(x+1)]\sin(\pi x)$ for the real-valued function $x(t)$.

(a) Find the fixed points of this differential equation. [Hint: there are infinitely many!]

(b)



Redraw the above sketch of $f(x)$ as a function of x for $x \in [-4, 5]$, and mark on it the fixed points that you found in (a).

(c) From an analysis of your sketch, determine the stability of each of these fixed points, and show the flow of $x(t)$ near the fixed points in the sketch from (b).

Homework Problem 5: Stability analysis in three dimensions [4]
Points: (a)[1](E); (b)[2](E); (c)[1](E)

Consider the following autonomous differential equation:

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} x^{10} - y^{24} \\ 1 - x \\ -3z - 3 \end{pmatrix}.$$

(a) Find the fixed points of this equation. [Check your result: for all fixed points, $\|\mathbf{x}^*\| = \sqrt{3}$.]

(b) Show that the fixed points are in general unstable, but that they are stable to deviations in certain directions. Determine the linear approximation to this equation for small deviations about the fixed point, and calculate the eigenvalues and eigenvectors of the corresponding matrix, A. [Check your results: for all fixed points, $|\det A| = 72$. Some of the eigenvalues for these fixed points are 6, 4, 12, -2.]

(c) Identify the stable directions, and the respective characteristic timescale for each deviation from the fixed point to decay to zero.

Homework Problem 6: Field lines of an electric quadrupole field in two dimensions [Bonus]
Points: [2](E,Bonus)

Consider the field $\mathbf{E} = F(x, -3z)^T$ in the xz-plane, generated by an electric quadrupole. The constant F governs the field strength. The shape of the field lines can be described by expressing z as function of x. Find a formula for $z(x)$ by solving a suitable differential equation. Sketch some field lines in each quadrant of the xz-plane to illustrate their shape. [Check your results: a field line passing through the point $(x, z)^T = (2, 1)^T$ also passes through $(1, 8)^T$.]

[Total Points for Homework Problems: 18]
