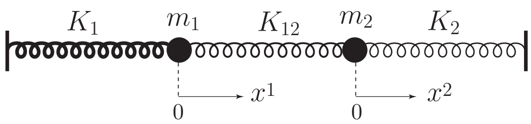
Sheet 12.2: Fourier Integrals, Differential Equations

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 3, 4, 5.
Videos exist for example problems 2 (C6.3.3), 3 (C7.5.1).

Optional Problem 1: Coupled oscillations of two point masses [5]
Points: (a)[0.5](E); (b)[0.5](E); (c)[2](E); (d)[2](M).

Consider a system of two point masses, with masses m_1 and m_2, which are connected to two fixed walls and to each other by means of three springs (spring constants K_1, K_{12} and K_2) (see sketch). The equations of motion for both masses are

$$\begin{aligned} m_1\ddot{x}^1 &= -K_1x^1 - K_{12}(x^1 - x^2), \\ m_2\ddot{x}^2 &= -K_2x^2 - K_{12}(x^2 - x^1). \end{aligned}$$



(a) Bring the system of equations into the form $\ddot{\mathbf{x}}(t) = -A \cdot \mathbf{x}(t)$, with $\mathbf{x} = (x^1, x^2)^T$. What is the form of matrix A?

[Check your result: $\det A = [K_1K_2 + (K_1 + K_2)K_{12}]/(m_1m_2)$.]

(b) Using the ansatz $\mathbf{x}(t) = \mathbf{v}\cos(\omega t)$, this system of differential equations can be converted to an algebraic eigenvalue problem. Find the form of this eigenvalue problem.

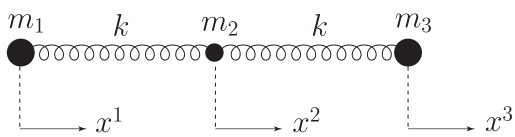
(c) Set $m_1 = m_2$, $K_2 = m_1\Omega^2$, $K_1 = 4K_2$ and $K_{12} = 2K_2$ (note that Ω has the dimension of frequency). Find the eigenvalues, λ_j, and the eigenvectors, $\mathbf{v}_j$, of the matrix $\frac{1}{\Omega^2}A$, and therefore the corresponding **eigenfrequencies**, ω_j, and **eigenmodes**, $\mathbf{x}_j(t)$, of the coupled masses (with $\mathbf{x}_j(0) = \mathbf{v}_j$). [Check your result: $\lambda_1 + \lambda_2 = 9$.]

(d) Make a sketch of both eigenmodes $\mathbf{x}_j(t)$ which shows both the $j = 1$ and 2 cases on the same set of axes. Comment on the physical behaviour that you observe!

Optional Problem 2: Coupled oscillations of three point masses [5]
Points: (a)[0.5](E); (b)[0.5](E); (c)[2](E); (d)[2](M)

Consider a system consisting of three masses, m_1, m_2 and m_3, coupled through two identical springs, each with spring constant k (see sketch). The equations of motion for the three masses read:

$$\begin{aligned} m_1\ddot{x}^1 &= -k(x^1 - x^2), \\ m_2\ddot{x}^2 &= -k\left([x^2 - x^1] - [x^3 - x^2]\right), \\ m_3\ddot{x}^3 &= -k(x^3 - x^2), \end{aligned}$$



(a) Bring this system of equations into the form $\ddot{\mathbf{x}}(t) = -A \cdot \mathbf{x}(t)$, with $\mathbf{x} = (x^1, x^2, x^3)^T$. What is the matrix A? [Check your result: $\det A = 0$.]

(b) By making the ansatz $\mathbf{x}(t) = \mathbf{v}\cos(\omega t)$, this system of equations can be reduced to an algebraic eigenvalue problem. Find this eigenvalue equation.

(c) From now on, set $m_1 = m_3 = m$, $m_2 = \frac{2}{3}m$, and $k = m\Omega^2$. (Ω has the dimension of a frequency.) Find the eigenvalues, λ_j, and normalized eigenvectors, $\mathbf{v}_j$, of the matrix $\frac{1}{\Omega^2}A$, and thus the corresponding eigenfrequencies, ω_j, and eigenmodes, $\mathbf{x}_j(t)$, of the coupled masses (with $\mathbf{x}_j(0) = \mathbf{v}_j$). [Check your result: $\lambda_1 + \lambda_2 + \lambda_3 = 5$.]

(d) Sketch the three eigenmodes $\mathbf{x}_j(t)$ as functions of time: for each $j = 1$, 2 and 3, make a separate sketch that displays the three components, $x_j^1(t)$, $x_j^2(t)$ and $x_j^3(t)$, on the same axis. Comment on the physical behaviour that you observe!

[Total Points for Optional Problems: 10]
