Sheet 11: Delta Function and Fourier Series

Posted: Mo 09.01.23 Central Tutorial: Th 12.01.23 Due: Th 19.01.23, 14:00
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 1, 3(a), 4, 5.
Videos exist for example problems 4 (C6.2.1), 5 (C6.3.5).

Example Problem 1: Integrals with δ function [3]
Points: (a)[0.5](E); (b)[0.5](E); (c)[0.5](M); (d)[1](M); (e)[0.1](E).

Calculate the following integrals (with $a \in \mathbb{R}$):

(a) $I_1(a) = \int_{-\infty}^{\infty} \mathrm{d}x \, \delta(x-\pi) \sin(ax)$

(b) $I_2(a) = \int_{\mathbb{R}^3} \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \, \delta(\mathbf{x}-\mathbf{y}) \, \|\mathbf{x}\|^2 \,,$ with $\mathbf{y} = (a, 1, 2)^T$

(c) $I_3(a) = \int_0^a \mathrm{d}x \, \delta(x-\pi) \dfrac{1}{a + \cos^2(x/2)}$

(d) $I_4(a) = \int_0^3 \mathrm{d}x \, \delta(x^2 - 6x + 8) \sqrt{\mathrm{e}^{ax}}$

(e) $I_5(a) = \int_{\mathbb{R}^2} \mathrm{d}x^1 \mathrm{d}x^2 \, \delta(\mathbf{x} - a\mathbf{y}) \, \mathbf{x} \cdot \mathbf{y} \,,$ with $\mathbf{y} = (1, 3)^T$. Remark: $\delta(\mathbf{x}) = \delta(x^1)\delta(x^2)$.

[Check your results: $I_1(\frac{1}{2}) = 1$, $I_2(1) = 6$, $I_3(\pi) = \frac{1}{2\pi}$, $I_4(\ln 2) = 1$, $I_5(1) = 10$.]

Example Problem 2: Lorentz representation of the Dirac δ-function [4]
Points: [4](M).

Explain why in the limit $\epsilon \to 0^+$, the Lorentz peak function $\delta^\epsilon(x)$ given below is a representation of the Dirac delta function $\delta(x)$. To this end, compute (i) the height, (ii) the width x_w (defined by $\delta^\epsilon(x_\mathrm{w}) = \frac{1}{2}\delta^\epsilon(0)$, $x_\mathrm{w} > 0$) and (iii) the area of the peak. How do these quantities behave for $\epsilon \to 0^+$? Furthermore, calculate the functions (iv) $\Theta^\epsilon(x) = \int_{-\infty}^x \mathrm{d}x' \delta^\epsilon(x')$ and (v) $\delta'^\epsilon(x) = \frac{\mathrm{d}}{\mathrm{d}x}\delta^\epsilon(x)$. Sketch Θ^ϵ, $\epsilon\delta^\epsilon$ and $\epsilon^2\delta'^\epsilon$ as functions of x/ϵ in three separate sketches (one beneath the other, with aligned y-axes and the same scaling for the x/ϵ-axes).

Lorentz-Peak: $\delta^\epsilon(x) = \dfrac{\epsilon/\pi}{x^2 + \epsilon^2}\,.$

Hint: When calculating the peak weight, use the substitution $x = \epsilon \tan y$.
Remark: Lorentzian functions are common in physics. Example: the energy spectrum of a discrete quantum state, which is weakly coupled to the environment, has the form of a Lorentzian function, the width of which is determined by the strength of the coupling to the environment. As the coupling strength approaches zero, we obtain a δ peak.

Example Problem 3: Series representation of hyperbolic functions [3]
Points: [3](E).

Compute the following series for $y \in \mathbb{R}^+$, by expressing each as a geometric series in $\omega \equiv \mathrm{e}^{-y}$.

(a) $\displaystyle\sum_{n=0}^{\infty} \mathrm{e}^{-y(n+1/2)}$, (b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \mathrm{e}^{-y(n+1/2)}$, (c) $\displaystyle\sum_{n\in\mathbb{Z}} \mathrm{e}^{-y|n|}$.

Example Problem 4: Fourier series of the sawtooth function [2]
Points: [2](M).

Let $f(x)$ be a sawtooth function, defined by $f(x) = x$ for $-\pi < x < \pi$, $f(\pm\pi) = 0$ and $f(x+2\pi) = f(x)$. Calculate the Fourier coefficients $\tilde{f}_n$ in the representation $f(x) = \frac{1}{L}\sum_n \mathrm{e}^{\mathrm{i}k_n x}\tilde{f}_n$. How should k_n and L be chosen? Sketch the function $f(x)$, as well as the sum of the $n = 1$ and $n = -1$ terms of the Fourier series (i.e. the first term of the corresponding sine series). [Check your result: $\tilde{f}_6 = \frac{1}{3}\mathrm{i}\pi$.]

Example Problem 5: Parseval's identity and convolution [7]
Points: (a)[3](M); (b)[2](M); (c)[2](M).

Let $f(x)$ be a sawtooth function, defined by $f(x) = x$ for $-\pi < x < \pi$, $f(\pm\pi) = 0$ and $f(x + 2\pi) = f(x)$. In the Fourier representation $f(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}nx}\tilde{f}_n$, its Fourier coefficients are $\tilde{f}_0 = 0$, $\tilde{f}_{n\neq 0} = 2\pi\mathrm{i}(-1)^n/n$. (See example problem 4.) Let $g(x) = \sin x$.

(a) Using this concrete example, check that Parseval's identity holds, by computing both the integral $\int_{-\pi}^{\pi} \mathrm{d}x\, \overline{f}(x)g(x)$ and the sum $(1/2\pi)\sum_n \overline{\tilde{f}}_n\, \tilde{g}_n$ explicitly.

(b) Prove the famous identity $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, by computing the integral $\int_{-\pi}^{\pi} \mathrm{d}x\, f^2(x)$ in two ways: first, by direct integration, and second, by expressing it as a sum over Fourier modes using Parseval's identity.

(c) Calculate the convolution $(f * g)(x)$ both by directly computing the convolution integral and by using the convolution theorem and a summation of Fourier coefficients.

[Total Points for Example Problems: 19]

Homework Problem 1: Integrals with δ function [4]
Points: (a)[0.5](E); (b)[0.5](E); (c)[0.5](M); (d)[1](M); (e)[1](A); (f)[0.5](E).

Calculate the following integrals (with $a \in \mathbb{R}$, $n \in \mathbb{N}$):

(a) $I_1(a) = \int_1^4 \mathrm{d}x\, \delta(x-2)\,(a^x + 3)$

(b) $I_2(a) = \int_{\mathbb{R}^2} \mathrm{d}x^1 \mathrm{d}x^2\, \delta(\mathbf{x}-\mathbf{y})\,(x^1 + x^2)^2\, \mathrm{e}^{3-x^1}$, with $\mathbf{y} = (3, a)^T$

(c) $I_3(a) = \int_{-1}^{1} \mathrm{d}x\, \sqrt{2+2x}\; \delta(ax-2)$, with $a \neq 0$

(d) $I_4(a) = \int_{-\infty}^{\infty} \mathrm{d}x\, \delta(3^{-x} - 9)(1 - x^a)$

(e) $I_5(n) = \int_{-\pi/2}^{9\pi/2} \mathrm{d}x\, \cos(nx)\; \delta\left(\sin x\right)$

(f) $I_6(a) = \int_{\mathbb{R}^2} \mathrm{d}x^1 \mathrm{d}x^2\, \delta(\mathbf{x}-\mathbf{y}) \mathrm{e}^{\|\mathbf{x}\|^2}$, with $\mathbf{y} = (a, -a)^T$

[Check your results: $I_1(3) = 12$, $I_2(-5) = 4$, $I_3(2) = \frac{1}{2}$, $I_4(3) = \frac{1}{\ln 3}$, $I_5(7) = 1$, $I_6(\frac{1}{\sqrt{2}}) = \mathrm{e}$.]

Homework Problem 2: Representations of the Dirac δ-function [4]
Points: [4](M).

Explain why in the limit $\epsilon \to 0^+$, the peak-shaped function $\delta^\epsilon(x)$ given below is a representation of the Dirac delta function $\delta(x)$. To this end, compute (i) the height, (ii) the width x_w (defined by $\delta^\epsilon(x_\mathrm{w}) = \frac{1}{2}\delta^\epsilon(0)$, $x_\mathrm{w} > 0$) and (iii) the area of the peak. How do these quantities behave for $\epsilon \to 0^+$? Furthermore, calculate the functions (iv) $\Theta^\epsilon(x) = \int_{-\infty}^{x} \mathrm{d}x' \delta^\epsilon(x')$ and (v) $\delta'^\epsilon(x) = \frac{\mathrm{d}}{\mathrm{d}x}\delta^\epsilon(x)$. Sketch Θ^ϵ, $\epsilon\delta^\epsilon$ and $\epsilon^2\delta'^\epsilon$ as functions of x/ϵ in three separate sketches (one beneath the other, with aligned y-axes and the same scaling for the x/ϵ-axes).

Derivative of the Fermi function: $\delta^\epsilon(x) = \dfrac{1}{4\epsilon} \dfrac{1}{\cosh^2[x/(2\epsilon)]}$.

Hint: When calculating the peak weight, use the substitution $y = \tanh[x/(2\epsilon)]$.
Remark: In condensed matter physics and nuclear physics the function $\delta^\epsilon(x)$ plays an important role: it arises as the derivative of the so-called **Fermi function**, $f(E) = \frac{1}{\mathrm{e}^{E/k_\mathrm{B}T}+1} = \Theta^{k_\mathrm{B}T}(-E)$, with $-\frac{\mathrm{d}}{\mathrm{d}E} f(E) = \delta^{k_\mathrm{B}T}(E)$, where $f(E)$ is the occupation probability of a fermionic single-particle state with energy E as function of the system's temperature T (k_B is the so-called Boltzmann constant). In the limit of zero temperature, $T \to 0$, the derivative of the Fermi function reduces to a Dirac δ-function.

Homework Problem 3: Series representation of the periodic δ function [5]
Points: (a)[0.5](E); (b)[0.5](M); (c)[1.5](A); (d)[0.5](E); (e)[1](A); (f)[0.5](E); (g)[0.5](E)

Show that the function $\delta^\epsilon(x)$, defined by

$$\delta^\epsilon(x) = \frac{1}{L} \sum_k \mathrm{e}^{\mathrm{i}kx - \epsilon|k|} , \quad k = 2\pi n/L, \quad n \in \mathbb{Z} , \quad x, \epsilon, L \in \mathbb{R} , \quad 0 < \epsilon \ll L , \tag{1}$$

has the following properties:

(a) $\delta^\epsilon(x) = \delta^\epsilon(x+L)$. (2)

(b) $\displaystyle\int_{-L/2}^{L/2} \mathrm{d}x\, \delta^\epsilon(x) = 1$. *Hint:* Treat $k=0$ and $k \neq 0$ separately in $\sum_k$. (3)

(c) $$\delta^\epsilon(x) = \frac{1}{2L}\left[\frac{1+w}{1-w} + \frac{1+\overline{w}}{1-\overline{w}}\right] = \frac{1}{L}\frac{1-\mathrm{e}^{-4\pi\epsilon/L}}{1+\mathrm{e}^{-4\pi\epsilon/L} - 2\mathrm{e}^{-2\pi\epsilon/L}\cos(2\pi x/L)} \;, \qquad (4)$$

where $w = \mathrm{e}^{2\pi(\mathrm{i}x-\epsilon)/L}$ and $\overline{w} = \mathrm{e}^{2\pi(-\mathrm{i}x-\epsilon)/L}$.
Hint: Write out the sum in Eq. (1) as a geometric series in powers of w and $\overline{w}$.

(d) $\lim_{\epsilon\to 0} \delta^\epsilon(x) = 0$ for $x \neq mL$, with $m \in \mathbb{Z}$. *Hint:* Start from Eq. (4).

(e) $\delta^\epsilon(x) \simeq \dfrac{\epsilon/\pi}{\epsilon^2 + x^2}$ for $|x|/L \ll 1$ and $\epsilon/L \ll 1$.

Hint: Taylor expand the numerator in Eq. (4) up to first order in $\tilde{\epsilon} = 2\pi\epsilon/L$, and the denominator up to second order in $\tilde{\epsilon}$ and $\tilde{x} = 2\pi x/L$.

(f) Sketch the function $\delta^\epsilon(x)$ qualitatively for $\epsilon/L \ll 1$ and $x \in [-\frac{7}{2}L, \frac{7}{2}L]$.

(g) Deduce that in the limit of $\epsilon \to 0$, $\delta^\epsilon(x)$ represents a periodic δ function, with

$$\delta^0(x) = \frac{1}{L}\sum_k \mathrm{e}^{\mathrm{i}kx} = \sum_{m\in\mathbb{Z}} \delta(x - mL) \;.$$

Homework Problem 4: Fourier series [4]
Points: (a)[2](E); (b)[2](M)

Determine the Fourier series for the following periodic functions, i.e. calculate the Fourier coefficients $\tilde{f}_n$ in the representation $f(x) = \frac{1}{L}\sum_n \mathrm{e}^{\mathrm{i}k_n x}\tilde{f}_n$. How should k_n and L be chosen in each case? Sketch the functions first.

(a) $f(x) = |\sin x|$, (b) $f(x) = \begin{cases} 4x & \text{for} \quad -\pi \le x < 0\,, \\ 2x & \text{for} \quad 0 \le x < \pi\,, \end{cases}$ and $f(x+2\pi) = f(x)$.

[Check your results: (a) $\tilde{f}_3 = -\frac{2}{35}$, (b) $\tilde{f}_3 = \frac{2}{9}(2 - 9\mathrm{i}\pi)$.]

Homework Problem 5: Computing an infinite series using the convolution theorem [1]
Points: (a)[0.5](E); (b)[0.5](M); (c)[2](A,Bonus)

This problem illustrates how a complicated sum may be calculated explicitly using the convolution theorem.
Consider the periodic function $f_\gamma(t) = f_\gamma(0)\mathrm{e}^{\gamma t}$ for $t \in [0,\tau)$ and $f(t+\tau) = f(t)$, with $f_\gamma(0) = 1/(\mathrm{e}^{\gamma\tau} - 1)$. Take both γ and τ to be positive numbers, so that $f_{\pm\gamma}(0) \gtrless 0$.

(a) Consider a Fourier series representation of $f_\gamma(t)$ of the following form:

$$f_\gamma(t) = \frac{1}{\tau}\sum_{\omega_n} \mathrm{e}^{-\mathrm{i}\omega_n t}\tilde{f}_{\gamma,n}, \qquad \tilde{f}_{\gamma,n} = \int_0^\tau dt \mathrm{e}^{\mathrm{i}\omega_n t} f_\gamma(t), \quad \text{with} \quad \omega_n = 2\pi n/\tau, \quad n \in \mathbb{Z}.$$

Show that the Fourier coefficients are given by $\tilde{f}_{\gamma,n} = 1/(\mathrm{i}\omega_n + \gamma)$.

(b) Use this result and the convolution theorem to express the following series as a convolution of f_γ and $f_{-\gamma}$:

$$S(t) = \sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega_n t}}{\omega_n^2 + \gamma^2} = -\tau \int_0^\tau dt' f_\gamma\left(t - t'\right) f_{-\gamma}\left(t'\right) . \tag{5}$$

(c) Sketch the functions $f_\gamma(t - t')$ and $f_{-\gamma}(t')$ occurring in the convolution theorem as functions of t', for $t' \in [-\tau, 2\tau]$. Assume $0 \leq t \leq \tau$ and show that the convolution integral (5) is given by the following expression:

$$S(t) = \frac{\tau\big[\sinh\left(\gamma\left(t - \tau\right)\right) - \sinh\left(\gamma t\right)\big]}{2\gamma\big[1 - \cosh\left(\gamma\tau\right)\big]} .$$

Hint: The integral $\int_0^\tau \mathrm{d}t'$ involves an interval of t' values for which $t - t'$ lies outside of $[0, \tau)$. It is therefore advisable to split the integral into two parts, with $\int_0^t \mathrm{d}t'$ and $\int_t^\tau \mathrm{d}t'$.

[Total Points for Homework Problems: 18]
