Sheet 11: Delta Function and Fourier Series

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 1, 3(a), 4, 5.
Videos exist for example problems 4 (C6.2.1), 5 (C6.3.5).

Optional Problem 1: Cosine Series [4]
Points: (a)[1](E); (b)[1](E); (c)[2](M).

For the function $f : I \to \mathbb{C}$, $x \mapsto f(x)$, with $I = [-L/2, L/2]$, consider the Fourier series representation $f(x) = \frac{1}{L}\sum_k \mathrm{e}^{\mathrm{i}kx}\tilde{f}_k$, with $k = \frac{2\pi n}{L}$ and $n \in \mathbb{Z}$.

(a) Show that the Fourier coefficients are given by $\tilde{f}_k = \int_{-L/2}^{L/2} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx} f(x)$.

(b) Now let f be an even function, i.e. $f(x) = f(-x)$. Show that then the Fourier coefficients are given by $\tilde{f}_k = 2\int_0^{L/2} \mathrm{d}x\, \cos(kx) f(x)$, and furthermore, that $f(x)$ can be represented by a cosine series of the form $f(x) = \frac{1}{2}a_0 + \sum_{k>0} a_k \cos(kx)$, with $k = \frac{2\pi n}{L}$ and $n \in \mathbb{N}_0$. Find a_k, expressed through $\tilde{f}_k$.

(c) Now consider the following function: $f(x) = 1$ for $|x| < L/4$, $f(x) = -1$ for $L/4 < |x| < L/2$. Sketch it, and compute the coefficients $\tilde{f}_k$ and a_k of the corresponding Fourier and cosine series. [Check your result: if $k = \frac{2\pi}{L}$, then $a_k = \frac{4}{\pi}$ and $\tilde{f}_k = \frac{2L}{\pi}$.]

Optional Problem 2: Sine Series [3]
Points: (a)[1](E); (b)[2](M)

For the function $f : I \to \mathbb{C}$, $x \mapsto f(x)$, with $I = [-L/2, L/2]$, consider the Fourier series representation $f(x) = \frac{1}{L}\sum_k \mathrm{e}^{\mathrm{i}kx}\tilde{f}_k$, with $k = \frac{2\pi n}{L}$ and $n \in \mathbb{Z}$, with Fourier coefficients $\tilde{f}_k = \int_{-L/2}^{L/2} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx} f(x)$.

(a) Let f be an odd function, i.e. $f(x) = -f(-x)$. Show that then the Fourier coefficients are given by $\tilde{f}_k = -2\mathrm{i}\int_0^{L/2} \mathrm{d}x\, \sin(kx) f(x)$, and furthermore, that $f(x)$ can be represented by a sine series of the form $f(x) = \sum_{k>0} b_k \sin(kx)$ with $k = \frac{2\pi n}{L}$ and $n \in \mathbb{N}_0$. What does b_k look like when expressed through $\tilde{f}_k$?

(b) Now consider the following function: $f(x) = 1$ for $0 < x < L/2$, $f(x) = -1$ for $-L/2 < x < 0$. Sketch it, and compute the coefficients $\tilde{f}_k$ and b_k of the corresponding Fourier and sine series. [Check your result: if $k = \frac{2\pi}{L}$, then $b_k = \frac{4}{\pi}$ and $\tilde{f}_k = \frac{2L}{i\pi}$.]

[Total Points for Optional Problems: 7]
