Sheet 10: Differential Equations II. Asymptotic Expansions

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 2, 3, 4, 5.
Videos exist for example problems 3 (C7.4.7), 5 (C5.4.1).

Optional Problem 1: Series expansion of inverse function [2]
Points: (a)[1](M); (b)[1](M).

This problem illustrates how the series expansion of an inverse function can be computed by expansion of the equation defining the inverse function.

The inverse, $g(x)$, of the function $f(x)$ fulfills the defining equation $f(g(x)) = x$. To find the series expansion of the inverse function around some point x_0, we may use the ansatz $g(x_0+x) \equiv y(x) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} y_n x^n$, and determine the coefficients $y_n \equiv y^{(n)}(0)$ by iteratively solving the equation $f(y(x)) = x_0 + x$ for $y(x)$. In this manner, calculate the series expansion of the following functions around $x = 0$, up to and including second order in x:

(a) $\ln(1+x)$, (b) 2^x.

[Check your results: (a) $y_2 = -1$, (b) $y_2 = \ln^2(2)$.]

Optional Problem 2: Series expansion of inverse function [2]
Points: (a)[1](M); (b)[1](M)

Find the series expansion of $\arcsin(x)$ around $x = 0$, up to and including order three, using both of the following methods:

(a) Find the expansion of $\arcsin(x) \equiv y(x)$ by iteratively solving the equation $\sin[y(x)] = x$.

(b) Since the sine function is odd, so is its inverse, hence it can be represented by the ansatz $\arcsin(x) = c_1 x^1 + \frac{1}{3!} c_3 x^3 + \mathcal{O}(x^5)$. Determine the coefficients c_1 and c_3 by expanding the equation $\arcsin(\sin(y)) = y$ in powers of y, using the known series expansion for $\sin(y)$. [Check your results: $c_3 = 1$.]

Optional Problem 3: Entropy maximization subject to constraints [2]

This problem and the next illustrate the use of Lagrange multipliers for a textbook topic from quantum statistical physics. For an in-depth discussion of the concepts mentioned below, refer to lecture courses in quantum physics and statistical physics.
Suppose a quantum system can be in any one of M possible states, $j = 1, \ldots, M$, with a probability p_j of being in the state j. The sum of these probabilities, $P = \sum_j p_j$, is fixed at $P = 1$. (Here, and in the following, $\sum_j$ stands for $\sum_{j=1}^{M}$.) When the system is in the quantum state j, the system has

energy E_j and particle number N_j. In quantum statistical physics, the **entropy**, S, and **average energy**, E, of the system are defined as:

$$S = -\sum_j p_j \ln p_j \ , \qquad E = \sum_j E_j p_j \ . \tag{1}$$

Show that maximizing the entropy $S(\{p_j\})$ with respect to the probabilities p_j, subject to the constraints set out below, leads to the following forms for the p_j's:

(a) If $P = 1$ is the only constraint, the entropy is maximal when all probabilities are equal, i.e. $p_j = 1/M$.

(b) If the constraint $P = 1$ is augmented by a second constraint, namely that the average energy has a specified value, $E = \sum_j E_j p_j$, the entropy is maximal when the probabilities p_j depend exponentially on the energies E_j as $p_j = Z^{-1} e^{-\beta E_j}$ (this is the Boltzmann distribution), where $Z = \sum_j e^{-\beta E_j}$ and $\beta > 0$ is a real constant.

Remarks: Z is known as the **partition function** of the system. In statistical physics, it is known that β is inversely proportional to the temperature, $\beta = 1/(k_{\mathrm{B}} T)$, where the **Boltzmann constant**, k_{B}, is a universal constant. The average energy of the system, given by $E = \sum_j E_j p_j = \sum_j E_j e^{-\beta E_j}/Z$, is therefore governed by temperature: when T increases, E increases as well. In the limit $T \gg \max(E_j)$ we have $p_j = 1/M$, just as in (a), i.e. then all states are equally likely. In the limit of $T = 0$, p_j is non-zero only if E_j equals the lowest energy in the spectrum. If there is only a single state with lowest energy (a 'non-degenerate ground state'), say with index $i = 1$, we have $p_j = \delta_{i1}$, i.e. at zero temperature the system is in the ground state with certainty.

Optional Problem 4: Entropy maximization subject to constraints, continued [2]

Consider the same setup as in the previous problem. Show that maximizing the entropy with respect to the probabilities p_j, subject to the three constraints of $P = 1$, specified average energy $E = \sum_j p_j E_j$, and specified average particle number, $N = \sum_j p_j N_j$, leads to probabilities of the form $p_j = Z^{-1} e^{-\beta(E_j - \mu N_j)}$, where $Z = \sum_j e^{-\beta(E_j - \mu N_j)}$ and $\beta > 0$ and μ are constants. Here Z is known as the **grand-canonical partition function**, and the constant μ, referred to as the **chemical potential**, regulates the average number of particles.

[Total Points for Optional Problems: 8]
