Sheet 09: Taylor Series. Differential Equations I

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 2, 3, 5.
Videos exist for example problems 4 (L8.3.1).

Optional Problem 1: Integration by partial fraction expansion [4]
Points: (a)[2](M); (b)[2](M).

A function f is called a **rational function** if it can be expressed as a ratio $f(x) = P(x)/Q(x)$ of two polynomials, P and Q. Integrals of rational functions can be computed using **partial fraction decomposition**, a procedure that expresses f as the sum of a polynomial (possibly with degree 0) and several ratios of polynomials with simpler denominators. To achieve this, the denominator Q is factorized into a product of polynomials, q_j, of lower degree, $Q(x) = \prod_j q_j(x)$, and the function f is written as $f(x) = \sum_j p_j(x)/q_j(x)$. The form of the polynomials p_j in the numerators is fixed uniquely by the form of the polynomials P and q_j. (Since a partial fraction decomposition starts with a common denominator and ends with a sum of rational functions, it is in a sense the inverse of the procedure of adding rational functions by finding a common denominator.) If a complete factorization of Q is used, this yields a decomposition of the integral $\int \mathrm{d}x f(x)$ into a sum of integrals that can be solved by elementary means. Here we illustrate the method using some simple examples; for a systematic treatment, consult textbooks on calculus.

Use partial fraction decomposition to compute the following integrals, for $z \in (0, 2)$:

(a) $I(z) = \int_0^z \mathrm{d}x \; \frac{3}{(x+1)(x-2)}$, (b) $I(z) = \int_0^z \mathrm{d}x \; \frac{3x}{(x+1)^2(x-2)}$.

[Check your results: (a) $I(3) = -\ln 8$, (b) $I(3) = -\ln 4 + \frac{3}{4}$.]

Optional Problem 2: Integration by partial fraction decomposition [2]
Points: (a)[2](M); (b)[2](M,Bonus).

Use partial fraction decomposition to compute the following integrals, for $z \in (0, 1)$:

(a) $I(z) = \int_0^z \mathrm{d}x \frac{x+2}{x^3 - 3x^2 - x + 3}$, (b) $I(z) = \int_0^z \mathrm{d}x \, \frac{4x-1}{(x+2)(x-1)^2}$.

[Check your results: (a) $I(\frac{1}{2}) = \frac{5}{8}\ln 5 - \frac{1}{2}\ln 3$, (b) $I(\frac{1}{2}) = 1 - \ln\left(\frac{5}{2}\right)$.]

Optional Problem 3: Relativistic dispersion relation [1]

According to the special theory of relativity, the energy E of a particle of mass m is related to its momentum p by the following formula (dispersion relation),

$$E(p) = \sqrt{m^2c^4 + p^2c^2},$$

where c is the speed of light. Calculate the first three nonzero terms of the Taylor series of $E(p)$ for small p, where m and c are positive constants. Which of the terms in this expansion are familiar from classical mechanics?
Hint: Write $E(p)$ in the form $E(p) = mc^2\sqrt{1+x}$, with $x = p^2/(m^2c^2)$, and expand in terms of x. Then rewrite the formula in terms of p again.

[Total Points for Optional Problems: 7]
