Sheet 08: Matrices III: Unitary, Orthogonal, Diagonalization

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 2, 5, 6.
Videos exist for example problems 2 (L7.3.1), 6 (C4.5.5).

Optional Problem 1: Inertia tensor [2]
Points: [2](M).

The inertia tensor of a rigid body composed of point masses is defined as

$$\widetilde{I}_{ij} = \sum_a m_a \widetilde{I}_{ij}(\mathbf{r}_a, \mathbf{r}_a) , \quad \text{with} \quad \widetilde{I}_{ij}(\mathbf{r}, \mathbf{r}') \equiv \delta_{ij} \mathbf{r} \cdot \mathbf{r}' - (\mathbf{e}_i \cdot \mathbf{r})(\mathbf{e}_j \cdot \mathbf{r}') ,$$

where m_a and $\mathbf{r}_a = (r^1_{\ a}, r^2_{\ a}, r^3_{\ a})^T$ are, respectively, the mass and position of point mass a. The eigenvalues of the inertia tensor are known as the rigid body's *moments of inertia*.
Consider a rigid body consisting of three point masses, $m_1 = 4$, $m_2 = M$ and $m_3 = 1$, at positions $\mathbf{r}_1 = (1, 0, 0)^T$, $\mathbf{r}_2 = (0, 1, 2)^T$ and $\mathbf{r}_3 = (0, 4, 1)^T$, respectively. Determine its inertia tensor $\widetilde{I}$ and moments of inertia as functions of M. (Eigenvectors are not required.) [Check your results: if $M = 5$, then $\lambda_1 = 42$, $\lambda_2 = 39$, $\lambda_3 = 11$.]
Remark: For the purposes of diagonalizing the inertia tensor, it may be viewed as a matrix.

Optional Problem 2: Inertia tensor [2]
Points: (a)[1](E); (b)[1](E).

Consider a rigid body consisting of two point masses, $m_1 = \frac{2}{3}$ and $m_2 = 3$, at positions $\mathbf{r}_1 = (2, 2, -1)^T$ and $\mathbf{r}_2 = \frac{1}{3}(2, -1, 2)^T$, respectively.

(a) Show that its inertia tensor has the following form: $\widetilde{I} = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{pmatrix}$.

(b) Find the moments of inertia (eigenvalues). (Eigenvectors need not be computed.) (*Hint:* One eigenvalue is $\lambda = 3$.)

Optional Problem 3: Determinant of a triangular matrix [2]

Show that the determinant of a lower triangular matrix is given by the product of its diagonal elements:

$$\det \begin{pmatrix} a^1_{\ 1} & 0 & 0 & \dots & 0 \\ a^2_{\ 1} & a^2_{\ 2} & 0 & \dots & 0 \\ a^3_{\ 1} & a^3_{\ 2} & a^3_{\ 3} & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ a^n_{\ 1} & a^n_{\ 2} & \dots & & a^n_{\ n} \end{pmatrix}$$

Why does this statement also hold for an upper triangular matrix?

[Total Points for Optional Problems: 6]
