Sheet 07: Matrices II: Inverse, Basis Transformation

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 2, 3, 4, 6.
Videos exist for example problems 1 (L5.4.1), 5 (V2.5.1). Also see tutorvideos on "basis transformations".

Optional Problem 1: Linear maps and basis transformations in $\mathbb{R}^3$ [6]
Points: (a)[1](M); (b)[1](E); (c)[1](E); (d)[1](M); (e)[1](M); (f)[1](M)

Consider the following three linear transformations in $\mathbb{R}^3$, with standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$:

A : Rotation about the third axis by the angle $\theta_3 = \frac{\pi}{4}$, in the right-hand positive direction. *Hint:* Use the compact notation $\cos\theta_3 = \sin\theta_3 = s$.

B : Dilation (stretching) of the first axis by the factor $s_1 = 3$;

C : Rotation about the second axis by the angle $\theta_2 = \frac{\pi}{2}$, in the right-hand positive direction.

Hint: To understand what 'right-hand positive' means, imagine wrapping your right hand around the axis of rotation, with your thumb pointing in the positive direction. Your other fingers will be curled in the direction of 'positive rotation'.

(a) Find the matrix representations (with respect to the standard basis) of A, B and C.

(b) What is the image, $\mathbf{y} = B\mathbf{x}$, of the vector $\mathbf{x} = (1, 1, 1)^T$ under the dilation B?

(c) What is the image, $\mathbf{z} = D\mathbf{x}$, of $\mathbf{x}$ under the composition of all three maps, $D = C \cdot B \cdot A$? [Check your result: $z^2 = \sqrt{2}$.]

(d) Now consider a new basis, $\{\mathbf{e}'_i\}$, defined by a rotation of the standard basis by A, i.e. $\mathbf{e}_j \overset{A}{\mapsto} \mathbf{e}'_j$. Draw the new and old basis vectors in the same figure. Find the transformation matrix $T = \{T^i{}_j\}$, and specify the matrix elements of the transformation between the old and the rotated bases using $\mathbf{e}_j = \mathbf{e}'_i T^i{}_j$.

(e) In the $\{\mathbf{e}'_i\}$ basis, let the vectors $\mathbf{x}$ and $\mathbf{y}$ considered above be represented by $\mathbf{x} = \mathbf{e}'_i x'^i$ and $\mathbf{y} = \mathbf{e}'_i y'^i$. Find the corresponding components $\mathbf{x}' = (x'^1, x'^2, x'^3)^T$ and $\mathbf{y}' = (y'^1, y'^2, y'^3)^T$. [Check your results: $x'^1 = \sqrt{2}$, $y'^3 = 1$.]

(f) Let B' denote the dilation B in the rotated basis. Find B' by the appropriate transformation of the matrix B, and use the result to calculate the image $\mathbf{y}'$ of $\mathbf{x}'$ under B'. [Does the result match that from (e)?]

Optional Problem 2: Linear maps and basis transformations in $\mathbb{R}^3$ [6]
Points: (a)[1](M); (b)[1](E); (c)[1](E); (d)[1](M); (e)[1](M); (f)[1](M)

Consider the following three linear transformations in $\mathbb{R}^3$, with standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$:

$A:$ Rotation around the first axis by the angle $\theta_1 = -\frac{\pi}{3}$ in the right-handed sense, i.e. a left-handed rotation. *Hint:* Use the compact notation $\cos\theta_1 = c$, $\sin\theta_1 = s$.

$B:$ Dilation of the first and second axes by the factors $s_1 = 2$ and $s_2 = 4$ respectively.

$C:$ A reflection in the 2,3-plane.

(a) Find the matrix representations (using the standard basis) of A, B, C. Which of these transformations commute with each other (i.e. for which pairs of matrices does $M_1 M_2 = M_2 M_1$)?

(b) What is the image, $\mathbf{y} = CA\mathbf{x}$, of the vector $\mathbf{x} = (1,1,1)^T$ under the transformation CA?

(c) Find the vector $\mathbf{z}$, whose image under the composition of all three transformations, $D = C \cdot B \cdot A$, gives $\mathbf{y}$. [Hint: $D^{-1} = A^{-1}B^{-1}C^{-1}$.] [Check your result: $z^3 = \frac{1}{16}(7 - 3\sqrt{3})$.]

(d) Now consider a new basis, $\{\mathbf{e}'_i\}$, defined by a rotation and reflection CA of the standard basis, $\mathbf{e}'_i \overset{CA}{\mapsto} \mathbf{e}_i$. [*Caution:* in the example problem the order was reversed!] Sketch the old and new bases in the same picture. [*Note:* The new basis vectors are a left handed system! Why?] Find the transformation matrix T, and specify the matrix elements of the transformation between the old and the new basis, with $\mathbf{e}_j = \mathbf{e}'_i T^i{}_j$.

(e) In the $\{\mathbf{e}'_i\}$-basis, let the vectors $\mathbf{z}$ and $\mathbf{y}$ considered above be represented by $\mathbf{z} = \mathbf{e}'_i z'^i$ and $\mathbf{y} = \mathbf{e}'_i y'^i$. Find the corresponding components $\mathbf{z}' = (z'^1, z'^2, z'^3)^T$ and $\mathbf{y}' = (y'^1, y'^2, y'^3)^T$. [Check your results: $z'^3 = \frac{1}{2}(1 - \sqrt{3})$, $y'^2 = \frac{1}{2}(-1 + \sqrt{3})$.]

(f) Let D' denote the representation of D in the new basis. Find D' by an appropriate transformation of the matrix D, and use the result to find the image $\mathbf{y}'$ of $\mathbf{z}'$ under D'. [Does the result match the one from (e)?].

Optional Problem 3: Matrix inversion [Bonus]
Points: (a)[1](E); (b)[1](M); (c)[1](E)

Let M_n be an $n \times n$ matrix with matrix elements $(M_n)^i{}_j = \delta^i{}_j m + \delta^1{}_j$, with $i, j = 1, \ldots, n$, and $m \in \mathbb{R}$, $m \notin \{0, -1\}$.

(a) Find the inverse matrices M_2^{-1} and M_3^{-1}. Verify in both cases that $M_n^{-1} M_n = \mathbb{1}$.

(b) Use the results from (a) to formulate an Ansatz for the form of the inverse matrix M_n^{-1} for arbitrary n. Check your Ansatz by calculating $M_n^{-1} M_n$.

(c) Give a compact formula for the matrix elements $(M_n^{-1})^i{}_j$. Check its validity by showing that $\sum_l (M_n^{-1})^i{}_l (M_n)^l{}_j = \delta^i{}_j$ holds, by explicitly performing the sum on l.

Optional Problem 4: Matrix inversion [Bonus]
Points: (a)[1](E); (b)[1](M); (c)[1](E)

Let M_n be an $n \times n$ matrix with matrix elements $(M_n)^i{}_j = m\,\delta^i{}_j + \delta^{i+1}{}_j$, with $i, j = 1, \ldots, n$, and $m \in \mathbb{R}$, $m \neq 0$.

(a) Find the inverse matrices M_2^{-1} and M_3^{-1}. Verify in both cases that $M_n^{-1} M_n = \mathbb{1}$.

(b) Use the results from (a) to formulate a guess at the form of the inverse matrix M_n^{-1} for arbitrary n. Check your guess by calculating $M_n^{-1} M_n$.

(c) Give a compact formula for the matrix elements $(M_n^{-1})^i{}_j$. Check its validity by showing that $\sum_l (M_n^{-1})^i{}_l (M_n)^l{}_j = \delta^i_j$ holds, by explicitly performing the sum over l.

Optional Problem 5: Lorentz transformation [5]
Points: (a)[1](M); (b)[2](M); (c)[2](M)

(a) In a two dimensional Euclidean space, the distance between two points $\mathbf{P}_1 = (x^1, y^1)$ and $\mathbf{P}_2 = (x^2, y^2)$ is given by $\Delta s^2 = \Delta x^2 + \Delta y^2$ with $\Delta x = x^2 - x^1$, $\Delta y = y^2 - y^1$. Show that the rotation $R(\varphi)$ of the following form leaves the distance Δs^2 invariant (i.e. unchanged):

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = R(\varphi) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(b) In a two dimensional Minkowski space, the distance between two events $\mathbf{P}_1 = (ct^1, x^1)$ and $\mathbf{P}_2 = (ct^2, x^2)$ is given by $\Delta s^2 = \Delta(ct)^2 - \Delta x^2$. Show that the pseudo-rotation $\Lambda(\vartheta)$ of the following form leaves the distance Δs^2 invariant:

$$\begin{pmatrix} ct \\ x \end{pmatrix} \mapsto \begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda(\vartheta) \begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh\vartheta & \sinh\vartheta \\ \sinh\vartheta & \cosh\vartheta \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

Hint: $\cosh^2\vartheta - \sinh^2\vartheta = 1$.

(c) The Lorentz transformation which you know from experimental physics, can be written in the following matrix form with $\beta = v/c$ and $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$:

$$\begin{pmatrix} ct \\ x \end{pmatrix} \Rightarrow \begin{pmatrix} ct' \\ x' \end{pmatrix} = \tilde{\Lambda}(v) \begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

Show that this form corresponds to a pseudo-rotation i.e. that the matrix $\tilde{\Lambda}(v)$ has the same form as the matrix $\Lambda(\vartheta)$ from part (b) of the question. What is the relation between ϑ and v?

[Total Points for Optional Problems: 17]
