Sheet 05: Multidimensional Integration II. Fields I

Posted: Mo 14.11.22 Central Tutorial: Th 17.11.22 Due: Th 24.11.22, 14:00
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 2, 4, 7, 5.
Videos exist for example problems 2 (C4.2.1).

Example Problem 1: Gaussian integrals [3]
Points: (a)[1](M); (b)[1](M); (c)[1](M)

(a) Show that the two-dimensional Gaussian integral $I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\, \mathrm{e}^{-(x^2+y^2)}$ has the value $I = \pi$. *Hint:* Use polar coordinates; the radial integral can be solved by substitution.

(b) Now calculate the one-dimensional Gaussian integral

$$I_0(a) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2} \quad (a \in \mathbb{R},\ a > 0).$$

Hint: $I = [I_0(1)]^2$. Explain why! [Check your result: $I_0(\pi) = 1$.]

(c) Compute the one-dimensional Gaussian integral with a linear term in the exponent:

$$I_1(a, b) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2+bx} \quad (a, b \in \mathbb{R}, a > 0).$$

Hint: Write the exponent in the form $-ax^2 + bx = -a(x - C)^2 + D$ (called **completing the square**), then substitute $y = x - C$ and use the result from (b).

[Check your result: $I_1(1, 2) = \sqrt{\pi}\mathrm{e}$.]

Example Problem 2: Area of an ellipse (generalized polar coordinates) [2]
Points: (a)[1](M); (b)[1](E)

(a) Given a function $f : \mathbb{R} \to \mathbb{R}$ and two positive real numbers a, b, consider the two-dimensional integral of $f\big((x/a)^2 + (y/b)^2\big)$ over all $(x, y) \in \mathbb{R}^2$. Show that it can be written as

$$I = \int_{\mathbb{R}^2} \mathrm{d}x\, \mathrm{d}y\, f\big((x/a)^2 + (y/b)^2\big) = 2\pi ab \int_0^{\infty} \mathrm{d}\mu\, \mu\, f(\mu^2)\ ,$$

by transforming from Cartesian coordinates to generalized polar coordinates, defined as follows:

$$x = \mu a \cos\phi, \qquad y = \mu b \sin\phi\ ,$$

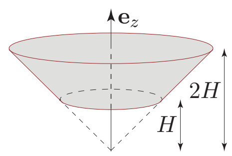
$$\mu^2 = (x/a)^2 + (y/b)^2, \qquad \phi = \arctan(ay/bx)\ .$$

Hint: For $a = b = 1$, they correspond to polar coordinates. For $a \neq b$, the local basis is *not* orthogonal!

(b) Using a suitable function f, calculate the area of an ellipse with semi-axes a and b, defined by $(x/a)^2 + (y/b)^2 \leq 1$.

Example Problem 3: Volume and moment of inertia (cylindrical coordinates) [2]
Points: (a)[1](E); (b)[1](E)

The moment of inertia of a rigid body with respect to a given axis of rotation is defined as $I = \int_V \mathrm{d}V\, \rho_0(\mathbf{r}) d_\perp^2(\mathbf{r})$, where $\rho_0(\mathbf{r})$ is the density at the point $\mathbf{r}$, and $d_\perp(\mathbf{r})$ the perpendicular distance from $\mathbf{r}$ to the rotation axis.



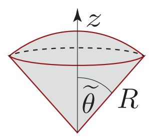
Let $F = \{\mathbf{r} \in \mathbb{R}^3 \,|\, H \leq z \leq 2H, \sqrt{x^2+y^2} \leq az\}$ be a homogeneous conical frustum (cone with tip removed) centered on the z-axis. Calculate, using cylindrical coordinates,

(a) its volume, $V_F(a)$, and

(b) its moment of inertia, $I_F(a)$, with respect to the z axis,

as functions of the dimensionless, positive scale factor a, the length parameter H, and the mass M of the frustum. [Check your results: $V_F(3) = 21\pi H^3$, $I_F(1) = \frac{93\pi}{70} M H^2$.]

Example Problem 4: Volume of a buoy (spherical coordinates) [2]
Points: (a)[1](E); (b)[1](M)



Consider a buoy, with its tip at the origin, bounded from above by a sphere centered on the origin, with $x^2 + y^2 + z^2 \leq R^2$, and from below by a cone with tip at the origin, with $z \geq a\sqrt{(x^2+y^2)}$.

(a) Show that the half angle at the tip of the cone is given by $\widetilde{\theta} = \arctan(1/a)$.

(b) Use spherical coordinates to calculate the volume $V(R, a)$ of the buoy as a function of R and a. [Check your results: $V(2, \sqrt{3}) = (16\pi/3)(1 - \sqrt{3}/2)$.]

Example Problem 5: Surface integral: area of a sphere [3]
Points: (a)[2](M); (b)[1](E)

Consider a sphere S with radius R. Compute its area, A_S, using (a) Cartesian coordinates, and (b) spherical coordinates, by proceeding as follows.

(a) Choose Cartesian coordinates, with the origin at the center of the sphere. Its area is twice that of the half-sphere S_+ lying above the xy-plane. S_+ can be parametrized as

$$\mathbf{r} : D \to S_+, \qquad (x, y)^T \mapsto \mathbf{r}(x, y) = (x, y, \sqrt{R^2 - x^2 - y^2})^T,$$

where $D = \{(x, y)^T \in \mathbb{R}^2 | x^2 + y^2 < R^2\}$ is a disk of radius R. Use this parametrization to compute the area of the sphere as $A_S = 2\int_D \mathrm{d}x\mathrm{d}y\, \|\partial_x \mathbf{r} \times \partial_y \mathbf{r}\|$.

(b) Now choose spherical coordinates and parametrize the sphere as

$$\mathbf{r} : U \to S, \qquad (\theta, \phi)^T \mapsto \mathbf{r}(\theta, \phi) = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T,$$

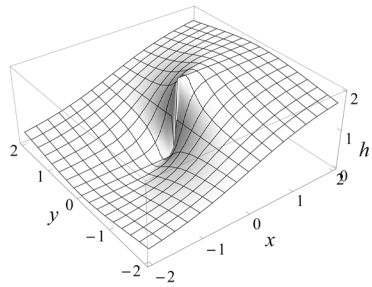
with $U = (0, \pi) \times (0, 2\pi)$. Compute its area, $A_S = \int_U \mathrm{d}\theta \mathrm{d}\phi \, \|\partial_\theta \mathbf{r} \times \partial_\phi \mathbf{r}\|$.

Example Problem 6: Gradient of $\mathrm{e}^{1/r}$ [1]
Points: [1](E)

Consider the scalar field $\varphi(\mathbf{r}) = \mathrm{e}^{1/r}$, where $r = \sqrt{x^2 + y^2 + z^2}$. At which spatial points does $|\boldsymbol{\nabla}\varphi| = \mathrm{e}$ hold?

Example Problem 7: Gradient of a mountainside [4]
Points: (a-h)[0.5 each](M)

A hiker encounters a mountainside (as shown in the figure) whose height is given by the function $h(\mathbf{r}) = \frac{x}{r} + 1$, with $\mathbf{r} = (x, y)^T$ and $r = \sqrt{x^2 + y^2}$. Describe the topography of the mountainside by answering the following questions. Make use of the properties of the gradient vector $\boldsymbol{\nabla} h_{\mathbf{r}}$.



(a) Calculate the gradient, $\boldsymbol{\nabla} h_{\mathbf{r}}$, and the total differential, $\mathrm{d}h_{\mathbf{r}}(\mathbf{n})$, for the vector $\mathbf{n} = (n_x, n_y)^T$.

(b) The hiker is at the point $\mathbf{r} = (x, y)^T$. In which direction does the mountainside increase most steeply?

(c) In which direction do the contour lines run at this point?

(d) Sketch a contour plot of the mountainside. Also draw the gradient vectors $\boldsymbol{\nabla} h_{\mathbf{r}}$ at the points $\mathbf{r}_1 = (-1, 1)^T$, $\mathbf{r}_2 = (0, \sqrt{2})^T$ and $\mathbf{r}_3 = (1, 1)^T$.

(e) Is there a contour line in the positive quadrant $(x, y \geq 0)$ such that $x = y$? If so, at what height does it occur?

(f) Find an equation describing the contour line at height $h(\mathbf{r}) = H$ in the positive quadrant $(x, y \geq 0)$.

(g) Where is the mountainside least steep? What is its height at that position?

(h) Where is the mountainside at its steepest? Describe, in detail, how its topography close to that point depends on x and y.

[Total Points for Example Problems: 17]

Homework Problem 1: Gaussian integrals [3]
Points: (a)[1](M); (b)[1](M); (c)[1](M)

Compute the following Gaussian integrals:

(a) $I_1(c) = \int_{-\infty}^{\infty} \mathrm{d}x \; \mathrm{e}^{-3(x+c)x}$ (b) $I_2(c) = \int_{-\infty}^{\infty} \mathrm{d}x \; \mathrm{e}^{-\frac{1}{2}\left(x^2+3x+\frac{c}{4}\right)}$

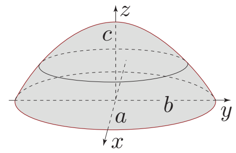
(c) $I_3(c) = \int_{-\infty}^{\infty} \mathrm{d}x \; \mathrm{e}^{-2(x+3)(x-c)}$

[Check your results: $I_1(2) = \sqrt{\frac{\pi}{3}}\mathrm{e}^3$, $I_2(1) = \sqrt{2\pi}\,\mathrm{e}$, $I_3(-3) = \sqrt{\frac{\pi}{2}}$.]

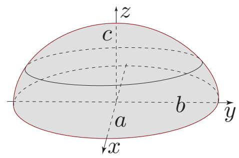
Homework Problem 2: Area integral for volume (generalized polar coordinates) [2]
Points: (a)[2](M); (b)[2](M,Bonus); (c)[0](A,Optional)

In the following, use generalized polar coordinates in two dimensions, defined as $x = \mu a \cos\phi$, $y = \mu b \sin\phi$, with $a, b \in \mathbb{R}$, $a > b > 0$. Calculate the volume $V(a,b,c)$ of the following objects T, E and C, as a function of the length parameters a, b and c.

(a) T is a tent with an elliptical base with semi-axes a and b. The height of its roof is described by the height function $h_T(x,y) = c\big[1 - (x/a)^2 - (y/b)^2\big]$.



(b) E is an ellipsoid with semi-axes a, b and c, defined by $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$.



(c) C is a cone with height c and an elliptical base with semi-axes a and b. All cross sections parallel to the base are elliptical, too. *Hint:* Augment the generalized polar coordinates by another coordinate, z (in analogy to passing from polar to cylindrical coordinates).

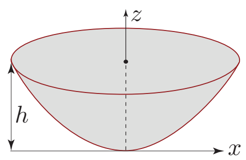
[Check your answers: if $a = 1/\pi$, $b = 2$, $c = 3$, then (a) $V_T = 3$, (b) $V_E = 8$, (c) $V_C = 2$.]

Homework Problem 3: Volume and moment of inertia (cylindrical coordinates) [4]
Points: (a)[0](M,Optional); (b)[4](M); (c)[3](A,Bonus)

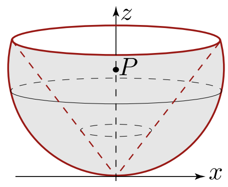
Consider the homogeneous rigid bodies C, P and B specified below, each with density ρ_0. For each body, use cylindrical coordinates to compute its volume, $V(a)$, and moment of inertia, $I(a) = \rho_0 \int_V \mathrm{d}V \, d_\perp^2$, with respect to the axis of symmetry, as functions of the dimensionless, positive scale factor a, the length parameter R, and the mass of the body, M.

(a) C is a hollow cylinder with inner radius R, outer radius aR, and height $2R$. [Check your results: $V_C(2) = 6\pi R^3$, $I_C(2) = \frac{15}{6} M R^2$.]

(b) P is a paraboloid with height $h = aR$ and curvature $1/R$, defined by $P = \{\mathbf{r} \in \mathbb{R}^3 \,|\, 0 \leq z \leq h, (x^2+y^2)/R \leq z\}$. [Check your results: $V_P(2) = 2\pi R^3$, $I_P(2) = \frac{2}{3} M R^2$.]



(c) B is the bowl obtained by taking a sphere, $S = \{\mathbf{r} \in \mathbb{R}^3 | \; x^2 + y^2 + (z - aR)^2 \le a^2R^2\}$, with radius aR, centered on the point $P: (0, 0, aR)^T$, and cutting a cone from it, $C = \{\mathbf{r} \in \mathbb{R}^3 | \, (x^2 + y^2) \le (a-1)z^2, a \ge 1\}$, which is symmetric about the z axis, with apex at the origin. [Check your results: $V_B\left(\frac{4}{3}\right) = \frac{16}{9}\pi R^3$, $I_B\left(\frac{4}{3}\right) = \frac{14}{15}MR^2$. What do you get for $a = 1$? Why?]



Hint: First, for a given z, find the radial integration boundaries, $\rho_1(z) \le \rho \le \rho_2(z)$, then the z integration boundaries, $0 \le z \le z_m$. What do you find for z_m, the maximal value of z?

Homework Problem 4: Volume integral over quarter sphere (spherical coordinates) [2]
Points: [2](M)

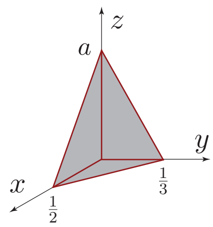
Use spherical coordinates to calculate the volume integral $F(R) = \int_Q \mathrm{d}V f(\mathbf{r})$ of the function $f(\mathbf{r}) = xy$ on the quarter sphere Q, defined by $x^2 + y^2 + z^2 \le R^2$ and $x, y \ge 0$. Sketch Q. [Check your result: $F(2) = \frac{64}{15}$.]

Homework Problem 5: Surface integral: area of slanted face of rectangular pyramid [2]
Points: [2](M).

Consider the pyramid shown in the sketch. Find a parametrization of its slanted face, F_{slant}, of the form

$$\mathbf{r} : U \subset \mathbb{R}^2 \to F_{\text{slant}} \subset \mathbb{R}^3, \quad (x, y)^T \mapsto \mathbf{r}(x, y),$$

i.e. specify the domain U and the Cartesian vector $\mathbf{r}(x, y)$. Then compute the area of the slanted face as $A_{\text{slant}} = \int_U \mathrm{d}x\mathrm{d}y \, \|\partial_x \mathbf{r} \times \partial_y \mathbf{r}\|$.
[Check your result: if $a = 2$, then $A_{\text{slant}} = \frac{\sqrt{53}}{12}$.]



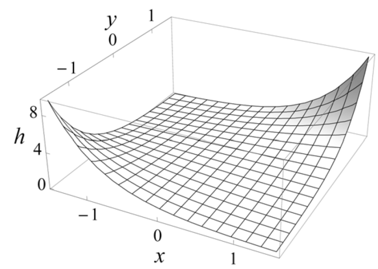
Homework Problem 6: Gradient of $\varphi(r)$ [2]
Points: (a)[1](E); (b)[1](E)

(a) For $\mathbf{r} \in \mathbb{R}^3$ and $r = \sqrt{x^2 + y^2 + z^2} = \|\mathbf{r}\|$, compute $\boldsymbol{\nabla} r$ and $\boldsymbol{\nabla} r^2$.

(b) Let $\varphi(r)$ be a general, twice differentiable function of r. Calculate $\boldsymbol{\nabla}\varphi(r)$ in terms of $\varphi'(r)$, the first derivative of φ with respect to r.

Homework Problem 7: Gradient of a valley [4]
Points: (a-f)[0.5 each](M); (g)[1](M)

A hiker encounters a valley as shown in the figure. The height of the valley is described by the equation $h(\mathbf{r}) = \mathrm{e}^{xy}$, with $\mathbf{r} = (x, y)^T$. Describe the topography of the valley by answering the following questions. Make use of the properties of the gradient vector $\boldsymbol{\nabla} h_{\mathbf{r}}$.



(a) Calculate the gradient $\boldsymbol{\nabla} h_{\mathbf{r}}$ and the total differential $\mathrm{d}h_{\mathbf{r}}(\mathbf{n})$ for the vector $\mathbf{n} = (n_x, n_y)^T$.

(b) The hiker stands at the point $\mathbf{r} = (x, y)^T$. In which direction does the slope of the valley increase most steeply?

(c) In which direction do the contour lines run at this point?

(d) Sketch a figure containing the contour plot of the side of the valley. Also draw the gradient vectors $\boldsymbol{\nabla} h_{\mathbf{r}}$ at the points $\mathbf{r}_1 = \frac{1}{\sqrt{2}}(-1, 1)^T$, $\mathbf{r}_2 = (0, 1)^T$ and $\mathbf{r}_3 = \frac{1}{\sqrt{2}}(1, 1)^T$.

(e) Obtain an equation for the contour line at a height $h(\mathbf{r}) = H(> 0)$.

(f) At what point is the valley least steep? What is its height at this point?

(g) At a distance of $r = \|\mathbf{r}\|$ from the origin, where is the valley at its steepest?

[Total Points for Homework Problems: 19]
