https://moodle.lmu.de → Kurse suchen: 'Rechenmethoden'

Sheet 05: Multidimensional Integration II. Fields I

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 2, 4, 7, 5.
Videos exist for example problems 2 (C4.2.1).

Optional Problem 1: Definite exponential integrals of the form $\int_0^\infty \mathrm{d}x\, x^n e^{-ax}$ [2]
Points: (a)[1](M); (b)[1](M)

Calculate the integral $I_n(a) = \int_0^\infty \mathrm{d}x\, x^n e^{-ax}$ (with $a \in \mathbb{R}$, $a > 0$, $n \in \mathbb{N}$) using two different methods: (a) repeated partial integration, and (b) repeated differentiation:

(a) Calculate I_0, I_1 and I_2 by using partial integration where necessary. Then use partial integration to show that

$$I_n(a) = \frac{n}{a} I_{n-1}(a)$$

for all $n \geq 1$. Use this relation iteratively to determine $I_n(a)$ as a function of a and n. $\big[$Check your result: $I_3(2) = \frac{3}{8}$.$\big]$

(b) Show that taking n derivatives of $I_0(a)$ with respect to a yields

$$I_n(a) = (-1)^n \frac{\mathrm{d}^n I_0(a)}{\mathrm{d}a^n}.$$

Then calculate these derivatives for a few small values of n. From the emerging pattern, deduce the general formula for $I_n(a)$.

Optional Problem 2: General Gaussian integrals [2]
Points: (a)[1](M); (b)[1](M)

Determine the value of the x^{2n} Gaussian integral, $I_n(a) = \int_{-\infty}^\infty \mathrm{d}x\, x^{2n}\, \mathrm{e}^{-ax^2}$ (with $a \in \mathbb{R}$, $a > 0$, $n \in \mathbb{N}$), using two different methods: (a) repeated partial integration, and (b) repeated differentiation:

(a) Starting from the Gaussian integral $I_0(a) = \sqrt{\frac{\pi}{a}}$, compute the integrals I_1 and I_2 by using partial integration where necessary. Then use partial integration to show that

$$I_n(a) = \frac{2n-1}{2a} I_{n-1}(a)$$

holds for all $n \geq 1$. Use this relation iteratively to determine $I_n(a)$ as a function of a and n. $\big[$ Check your result: $I_3(3) = \sqrt{\frac{\pi}{3}}\frac{5}{72}$.$\big]$

(b) Show that taking n derivatives of $I_0(a)$ with respect to a yields

$$I_n(a) = (-1)^n \frac{\mathrm{d}^n I_0(a)}{\mathrm{d}a^n}.$$

Then calculate these derivatives for a few small values of n. From the emerging pattern, deduce the general formula for $I_n(a)$.

Optional Problem 3: Volume and surface integral: parabolic solid of revolution [3]
Points: (a)[1](E); (b)[2](M).

Consider a parabolic solid of revolution, P, bounded from above by the plane $z = z_{\max}$, and from below by the surface of revolution obtained by rotating the parabola $z(x) = x^2$ about the z-axis.

(a) Calculate the volume, V, of the body P.

(b) Calculate the surface area, A, of the curved part, C, of the surface of P.

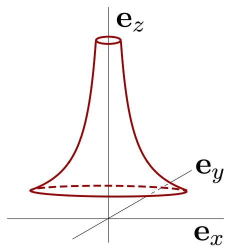
[Check your results: For $z_{\text{max}} = \frac{3}{4}$ we have $V = \frac{9\pi}{32}$ and $A = \frac{7\pi}{6}$.]

Optional Problem 4: Surface integral: hyperbolic solid of revolution (Gabriel's horn) [4]

Points: (a)[1](E); (b)[2](A); (c)[0.5](E); (d)[0.5](E).

Consider the solid body, K, generated by rotating the function $\rho(z) = 1/z$, with $1 \leq z \leq a$, about the z-axis. This shape is known as Gabriel's horn or Torticelli's trumpet.

(a) Compute the volume, $V(a)$, of the body K.
[Check your result: $V(2) = \frac{\pi}{2}$.]



(b) Write down the integral for the surface area of this solid, $A(a)$, and calculate its derivative, $A'(a) = \frac{\mathrm{d}}{\mathrm{d}a} A(a)$. [Check your result: $A'(1) = 2\sqrt{2}\pi$.]

(c) Find a lower bound for the value of the integral $A(a)$ by using the inequality $\sqrt{z^{-4} + 1} \geq 1$.

(d) How large are the volume and (the lower bound for) the area in the limit as $a \to \infty$?

Optional Problem 5: Area of a circular cone [2]
Points: [2](M)

Consider a circular cone, C, of radius R and height h. Compute the area, $A_C(R, h)$, of its (slanted) conical surface S_C as a function of R and h. [Check your result: $A_C(3, 4) = 15\pi$.]

Optional Problem 6: Area of an elliptical cone [2]
Points: [2](M)

Consider an elliptical cone, C, with semi-axes a and b and height h. Use generalized polar coordinates to show that the area, A_C, of its (slanted) conical surface S_C is given by an integral of the form,

$$A_C = \int_{S_C} \mathrm{d}S = P \int_0^{2\pi} \mathrm{d}\phi \, \sqrt{1 + Q \sin^2 \phi}\,,$$

and find $P(a,b,h)$ and $Q(a,b,h)$ as functions of a, b and h. *Remark*: This integral belongs to the class of so-called 'elliptical integrals', which cannot be solved in closed form.
[Check your results: if $a = 3$, $b = 2$ and $h = 4$, then $P = 5$ and $Q = \frac{4}{5}$.]

[Total Points for Optional Problems: 15]
