Sheet 04: Multidimensional Differentiation and Integration I

Posted: Mo 07.11.22 Central Tutorial: 10.11.22 Due: Th 17.11.22, 14:00
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 3, 4(a,b), 7(a-c), 9.
Videos exist for example problems 7 (V2.3.3), 8 (V2.3.5).

Example Problem 1: Partial derivatives [1]
Points: (a)[1](E); (b)[1](E,Bonus).

Compute the partial derivates $\partial_x f(x,y)$ and $\partial_y f(x,y)$ of the following functions:
[Check your results against those in square brackets.]

(a) $f(x,y) = x^2y^3 - 2xy$ $[\partial_x f(2,1) = 2, \quad \partial_y f(1,2) = 10]$

(b) $f(x,y) = \sin\big[x\mathrm{e}^{2y}\big]$ $\big[\partial_x f(0,\frac{1}{2}) = \mathrm{e}, \quad \partial_y f(\pi,0) = -2\pi\big]$

Example Problem 2: Chain rule for functions of two variables [2]
Points: [2](E)

This problem aims to illustrate the inner life of the chain rule for a function of several variables. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $\mathbf{y} = (y^1, y^2)^T \mapsto f(\mathbf{y}) = \|\mathbf{y}\|^2$ and the vector field $\mathbf{g} : \mathbb{R}^2_+ \to \mathbb{R}^2$, $\mathbf{x} = (x^1, x^2)^T \mapsto \mathbf{g}(\mathbf{x}) = (\ln x^2, 3\ln x^1)^T$, then $f(\mathbf{g}(\mathbf{x}))$ gives the norm of $\mathbf{g}$ as a function of $\mathbf{x}$. Find the partial derivatives $\partial_{x^1} f(\mathbf{g}(\mathbf{x}))$ and $\partial_{x^2} f(\mathbf{g}(\mathbf{x}))$ as functions of x^1 and x^2 in two ways,

(a) by first computing $f(\mathbf{x}) = f(\mathbf{g}(\mathbf{x}))$ as function of $\mathbf{x}$ and then taking partial derivatives;

(b) by using the chain rule $\partial_{x^k} f(\mathbf{g}(\mathbf{x})) = \sum_j \partial_{g^j} f(\mathbf{g}(\mathbf{x}))\, \partial_{x^k} g^j(\mathbf{x})$.

Why do both routes yield the same answer? Identify the similarities in both computations!
[Check your results: if $x^1 = 9$, $x^2 = 2$, then $\partial_{x^1} f = 4\ln 3$, $\partial_{x^2} f = \ln 2$.]

Example Problem 3: Two-dimensional integration (Cartesian coordinates) [Bonus]
Points: [2](M,Bonus)

Calculate the surface integral $I(a) = \int_{G_a} \mathrm{d}x\,\mathrm{d}y\, f(x,y)$ of the function $f(x,y) = xy$, over the area $G = \{(x,y) \in \mathbb{R}^2;\ 0 \le y \le 1;\ 1 \le x \le a - y\}$, with $2 \le a \in \mathbb{R}$.
[Check your result: $I(2) = \frac{5}{24}$].

Example Problem 4: Area enclosed by curves (Cartesian coordinates) [3]
Points: (a)[1](E); (b)[1](M); (c)[1](M)

Consider the curve $\gamma_1 : \mathbb{R} \to \mathbb{R}^2, t \mapsto (t, b(1 - t/a))^T$ and the closed curve $\gamma_2 : (0, 2\pi) \subset \mathbb{R} \to \mathbb{R}^2, t \mapsto (a\cos t, b\sin t)^T$ in Cartesian coordinates, with $0 < a, b \in \mathbb{R}$.

(a) Sketch the curves γ_1 and γ_2.

(b) Compute the area $S(a,b)$ enclosed by γ_2. [Check your result: $S(1,1) = \pi$.]

(c) γ_1 divides the area enclosed by γ_2 into two parts. Find the area $A(a,b)$ of the smaller part by computing an area integral. Check your result using elementary geometrical considerations.

Example Problem 5: Area integral for volume of a pyramid (Cartesian coordinates) [3]
Points: (a)[1](E); (b)[2](M)

Consider the pyramid bounded by the xy-plane, the yz-plane, the xz-plane and the plane $E = \{(x,y,z) \in \mathbb{R}^3, z = c(1 - x/a - y/b)\}$, with $0 < a, b, c \in \mathbb{R}$.

(a) Make a qualitative sketch of the pyramid. Find its volume $V(a,b,c)$ using geometric arguments [Check your result: $V(1,1,1) = \frac{1}{6}$.]

(b) Compute $V(a,b,c)$ by integrating the height $h(x,y)$ of the pyramid over its base area in the xy plane.

Example Problem 6: Coordinate transformations [3]
Points: [3](E)

Consider three points whose Cartesian coordinates, (x,y,z), are P_1: $(3,-2,4)$, P_2: $(1,1,1)$ and P_3: $(-3,0,-2)$. What is the representation of these three points in cylindrical coordinates, (ρ,ϕ,z), and in spherical coordinates, (r,θ,ϕ)? (Give the angles in radians.)

Example Problem 7: Cylindrical coordinates: velocity, kinetic energy, angular momentum [3]
Points: (a)[2](E); (b)[1](E,Bonus); (c)[1](E); (d)[0,5](E,Bonus); (e)[0,5](E,Bonus).

The relation between Cartesian and cylindrical coordinates is given by: $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$, with $\rho \in (0,\infty)$, $\phi \in (0,2\pi)$, $z \in (-\infty,\infty)$.
Basis vectors: Construct the basis vectors of the local basis for cylindrical coordinates, $\{\mathbf{e}_{y_i}\} = \{\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z\}$, and show explicitly that they have the following properties:
(a) $\mathbf{e}_{y_i} \cdot \mathbf{e}_{y_j} = \delta_{ij}$ and (b) $\mathbf{e}_{y_i} \times \mathbf{e}_{y_j} = \varepsilon_{ijk}\mathbf{e}_{y_k}$.
Physical quantities: Show that in cylindrical coordinates (c) the velocity vector, $\mathbf{v} = \frac{d}{dt}\mathbf{r}$, (d) the kinetic energy, $T = \frac{1}{2}m\mathbf{v}^2$, and (e) the angular momentum, $\mathbf{L} = m(\mathbf{r}\times\mathbf{v})$, have the following forms:

$$\begin{aligned}
\mathbf{v} &= \dot\rho\,\mathbf{e}_\rho + \rho\dot\phi\,\mathbf{e}_\phi + \dot z\,\mathbf{e}_z, \qquad T = \tfrac{1}{2}m\big[\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2\big], \\
\mathbf{L} &= m\big[-z\rho\dot\phi\,\mathbf{e}_\rho + (z\dot\rho - \rho\dot z)\,\mathbf{e}_\phi + \rho^2\dot\phi\,\mathbf{e}_z\big].
\end{aligned}$$

Example Problem 8: Line integral in polar coordinates: spiral [2]
Points: (a)[2](E); (b)[1](E)

The curve $\gamma_S = \{\mathbf{r}(\rho,\phi) \in \mathbb{R}^2 |\ \rho = R + \frac{1}{2\pi}\phi\Delta,\ \phi \in (0,2\pi)\}$, with $0 < R, \Delta \in \mathbb{R}$, describes a spiral path in two dimensions, parametrized using polar coordinates.

(a) Sketch the spiral path γ_S and calculate the line integral $W_1[\gamma_S] = \int_{\gamma_S} \mathrm{d}\mathbf{r}\cdot\mathbf{F}_1$ of the field $\mathbf{F}_1 = \mathbf{e}_\phi$ along γ_S. [Check your result: if $R = \Delta = 1$, then $W_1[\gamma] = 3\pi$.]

(b) Calculate the line integral $W_2[\gamma] = \int_\gamma \mathrm{d}\mathbf{r} \cdot \mathbf{F}_2$ of the field $\mathbf{F}_2 = \mathbf{e}_x$ along the straight path γ_G from the point $(R, 0)^T$ to the point $(R + \Delta, 0)^T$, and also along the spiral path γ_S. Are the results related? Explain!

Example Problem 9: Line integral in spherical coordinates: satellite in orbit [Bonus]
Points: (a)[0,5](E,Bonus); (b)[1](E,Bonus); (c)[0,5](E,Bonus); (d)[1](M,Bonus)

A satellite travels along an unusual trajectory γ that circles the north-south-axis of the earth as it travels from a point high above the north pole to a point high above the south pole. In spherical coordinates, the trajectory is given by $r(t) = r_0$, $\theta(t) = \omega_1 t$, $\phi(t) = \omega_2 t$, with $t \in (0, \pi/\omega_1)$. Due to the rotation of the earth, there is a wind in the upper atmosphere, exerting a force $\mathbf{F} = -F_0 \sin\theta \, \mathbf{e}_\phi$ on the satellite.

(a) Make a qualitative sketch of the orbit, for $\omega_2 = 20\omega_1$. How many times does the path circle around the north-south axis?

(b) What is the velocity vector $\dot{\mathbf{r}}$ written in spherical coordinates?

(c) Give the length $L[\gamma]$ of the orbit in terms of an integral. (You are not required to solve it.)

(d) Use the line integral $W[\gamma] = \int_\gamma \mathrm{d}\mathbf{r} \cdot \mathbf{F}$, to compute the work performed against the wind by the satellite along its orbit. [Check your result: if $F_0 = r_0 = \omega_1 = \omega_2 = 1$, then $W[\gamma] = -\frac{\pi}{2}$.]

[Total Points for Example Problems: 17]

Homework Problem 1: Partial derivatives [1]
Points: (a)[1](E); (b)[1](E,Bonus).

Compute the partial derivates $\partial_x f(x, y)$ and $\partial_y f(x, y)$ of the following functions:
[Check your results against those in square brackets.]

(a) $f(x, y) = \mathrm{e}^{-x^2 \cos(y)}$ $\qquad$ $\left[\partial_x f(1, \pi) = 2\mathrm{e}, \quad \partial_y f(1, \frac{\pi}{2}) = 1\right]$

(b) $f(x, y) = \sinh\left(\frac{x}{y}\right)$ $\qquad$ $\left[\partial_x f(\ln 2, 1) = \frac{5}{4}, \quad \partial_y f(\ln 2, 1) = -\frac{5}{4} \ln 2\right]$

Homework Problem 2: Chain rule for functions of two variables [2]
Points: [2](E)

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $\mathbf{y} = (y^1, y^2)^T \mapsto f(\mathbf{y}) = \mathbf{y} \cdot \mathbf{a}$, where $\mathbf{a} = (a^1, a^2)^T \in \mathbb{R}^2$, and the vector field $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{x} = (x^1, x^2)^T \mapsto \mathbf{g}(\mathbf{x}) = \mathbf{x}(\mathbf{x} \cdot \mathbf{b})$, where $\mathbf{b} = (b^1, b^2)^T \in \mathbb{R}^2$. Compute the partial derivatives $\partial_{x^k} f(\mathbf{g}(\mathbf{x}))$ (with $k = 1, 2$) as functions of $\mathbf{x}$,

(a) by first computing $f(\mathbf{g}(\mathbf{x}))$ explicitly and then taking partial derivatives;

(b) by using the chain rule $\partial_{x^k} f(\mathbf{g}(\mathbf{x})) = \sum_j \partial_{g^j} f(\mathbf{g}(\mathbf{x})) \partial_{x^k} g^j(\mathbf{x})$.

[Check your result: if $\mathbf{a} = (0, 1)^T$, $\mathbf{b} = (1, 0)^T$, then $\partial_{x^1} f(\mathbf{g}(\mathbf{x})) = x^2$, $\partial_{x^2} f(\mathbf{g}(\mathbf{x})) = x^1$.]
Hint: If compact notation is used, such as $\mathbf{a} \cdot \mathbf{x} = a_l x^l$ and $\partial_{x^k} x^l = \delta^l_k$, the computations are quite short.

Homework Problem 3: Two-dimensional integration (Cartesian coordinates) [Bonus]
Points: [2](M,Bonus)

Calculate the surface integral $I(a) = \int_G \mathrm{d}x\,\mathrm{d}y\, f(x,y)$ of the function $f(x,y) = y^2 + x^2$ over the surface $G = \{(x,y) \in \mathbb{R}^2;\ 0 \le x \le 1;\ 0 \le y \le \mathrm{e}^{ax}\}$, with $a \in \mathbb{R}$. *Hint:* for $\int \mathrm{d}x\, x^2 \mathrm{e}^{ax}$, use partial integration twice! [Check your result: $I(1) = \mathrm{e} + (\mathrm{e}^3 - 19)/9$.]

Homework Problem 4: Area enclosed by curves (Cartesian coordinates) [3]
Points: (a)[1](E); (b)[2](M)

Consider the curves $\gamma_1 : \mathbb{R} \to \mathbb{R}^2, t \mapsto ((t-2a)^2 + 2a^2, t)^T$ and $\gamma_2 : \mathbb{R} \to \mathbb{R}^2, t \mapsto (2(t-a)^2, t)^T$ in Cartesian coordinates, with $0 < a \in \mathbb{R}$.

(a) Sketch the curves γ_1 and γ_2.

(b) Compute the finite area $S(a)$ enclosed between these curves. [Check your result: $S(\frac{1}{2}) = \frac{4}{3}$.]

Homework Problem 5: Area integral for volume of ellipsoidal tent (Cartesian coordinates) [3]
Points: (a)[1](E); (b)[2](M)

A tent has a flat, ellipsoidal base, given by the equation $(x/a)^2 + (y/b)^2 \le 1$. The shape of the tent's roof is given by the height function $h(x,y) = c\big[1 - (x/a)^2 - (y/b)^2\big]$.

(a) Give a qualitative sketch of the shape of the tent, for $a = 2$, $b = 1$ and $c = 2$.

(b) Calculate the volume V of the tent via a surface integral of the height function. Use Cartesian coordinates. [Check your result: if $a = b = c = 1$, then $V = \pi/2$.]
Hint: Show by a suitable trigonometric substitution that $\int_0^1 \mathrm{d}x\,(1-x^2)^{3/2} = \frac{3}{16}\pi$.

Homework Problem 6: Coordinate transformations [2]
Points: [2](E)

The point P_1 has spherical coordinates $(r,\theta,\phi) = (2, \pi/6, 2\pi/3)$. What are its Cartesian and cylindrical coordinates, (x,y,z) and (ρ,ϕ,z), respectively? The point P_2 has cylindrical coordinates $(\rho,\phi,z) = (4, \pi/4, 2)$. What are its Cartesian and spherical coordinates? (Give the angles in radians.)

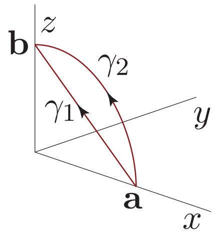
Homework Problem 7: Spherical coordinates: velocity, kinetic energy, angular momentum [3]
Points: (a)[2](E); (b)[1](E,Bonus); (c)[1](E); (d)[0,5](E,Bonus); (e)[0,5](E,Bonus)

The relationship between Cartesian and spherical coordinates is given by: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, with $r \in (0,\infty)$, $\phi \in (0,2\pi)$, $\theta \in (0,\pi)$.
Basis vectors: Construct the basis vectors of the local basis for spherical coordinates, $\{\mathbf{e}_{y_i}\} = \{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$, and show explicitly that
(a) $\mathbf{e}_{y_i} \cdot \mathbf{e}_{y_j} = \delta_{ij}$ and (b) $\mathbf{e}_{y_i} \times \mathbf{e}_{y_j} = \varepsilon_{ijk}\,\mathbf{e}_{y_k}$.
Physical quantities: Show that in spherical coordinates (c) the velocity vector, $\mathbf{v} = \frac{d}{dt}\mathbf{r}$, (d) the kinetic energy, $T = \frac{1}{2}m\mathbf{v}^2$, and (e) the angular momentum, $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$, have the following forms:

$$\mathbf{v} = \dot{r}\,\mathbf{e}_r + r\,\dot{\theta}\mathbf{e}_\theta + r\dot{\phi}\sin\theta\,\mathbf{e}_\phi, \quad T = \tfrac{1}{2}m\big[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\big], \quad \mathbf{L} = mr^2\big[\dot{\theta}\,\mathbf{e}_\phi - \dot{\phi}\sin\theta\,\mathbf{e}_\theta\big].$$

Homework Problem 8: Line integral in Cartesian and spherical coordinates [3]
Points: (a)[1](E); (b)[2](E)

Consider the vector field $\mathbf{F} = (0, 0, fz)^{\mathrm{T}}$, with $f \in \mathbb{R}$. Compute the line integral $W[\gamma] = \int_\gamma \mathrm{d}\mathbf{r} \cdot \mathbf{F}$ from $\mathbf{a} = (1, 0, 0)^T$ to $\mathbf{b} = (0, 0, 1)^T$ explicitly along the following two paths:



(a) γ_1: a straight line. [Check your result: if $f = 2$, then $W[\gamma_1] = 1$.]

(b) γ_2: a segment of a circle with radius $R = 1$ centered at the origin. Use spherical coordinates. [Check your result: if $f = 3$, then $W[\gamma_2] = \frac{3}{2}$.]

Homework Problem 9: Line integrals in cylindrical coordinates: bathtub drain [Bonus]
Points: (a)[0,5](E,Bonus); (b)[1](E,Bonus); (c)[0,5](M,Bonus); (d)[1](M,Bonus)

A soap bubble travels along a spiral-shaped path γ towards the drain of a bathtub. In cylindrical coordinates the path is given by $\rho(t) = \rho_0 \mathrm{e}^{-t/\tau}$, $\phi(t) = \omega t$, $z(t) = z_0 \mathrm{e}^{-t/\tau}$, with $\rho_0 > \rho_{\mathrm{d}}$ and $t \in [0, t_{\mathrm{d}}]$, where ρ_{d} is the drain radius and $t_{\mathrm{d}} = \tau \ln(\rho_0/\rho_{\mathrm{d}})$ the time at which the bubble reaches the drain.

(a) Make a qualitative sketch of the path (e.g. for $\omega = 6\pi/\tau$ in $\rho_0 = 10\rho_{\mathrm{A}}$).

(b) What is the velocity vector $\mathbf{v} = \dot{\mathbf{r}}$ in cylindrical coordinates? What is the magnitude of the final velocity, i.e $v_{\mathrm{d}} = \|\mathbf{v}(t_{\mathrm{d}})\|$?

(c) Show that the length of the path is given by $L[\gamma] = \tau v_{\mathrm{d}} \left(\rho_0/\rho_{\mathrm{d}} - 1\right)$.

(d) Using the line integral $W[\gamma] = \int_\gamma d\mathbf{r} \cdot \mathbf{F}$, find the work done by gravity $\mathbf{F} = -mg\mathbf{e}_z$ along the path of the soap bubble. Give a physical interpretation for this result!

[Check your results: if $\tau = 2/\omega$, $z_0 = 2\rho_0$ and $\rho_{\mathrm{d}} = \rho_0/3$, then: (b) $v_{\mathrm{d}} = \rho_0/\tau$, (c) $L = 2\rho_0$, (d) $W[\gamma] = mg\rho_0 4/3$.]

[Total Points for Homework Problems: 17]
