Sheet 04: Multidimensional Differentiation and Integration I

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 3, 4(a,b), 7(a-c), 9.
Videos exist for example problems 7 (V2.3.3), 8 (V2.3.5).

Optional Problem 1: Partial derivates of first and second order [2]
Points: [2](E)

Consider the function $f : \mathbb{R}^2\backslash(0,0)^T \to \mathbb{R}$, $\mathbf{r} = (x,y)^T \mapsto f(\mathbf{r}) = \frac{x}{r} + 1$, with $r = \sqrt{x^2+y^2}$. Calculate all possible partial derivatives of first and second order.

Optional Problem 2: Partial derivates of first and second order [2]
Points: [2](E)

Calculate all possible partial derivatives of first and second order of the function $f : \mathbb{R}^3 \to \mathbb{R}$, $\mathbf{r} = (x,y,z)^T \mapsto f(\mathbf{r})$, for $f(\mathbf{r}) = x^2\ln(y)/z$.

Optional Problem 3: Fubini's theorem [2]
Points: [2](M)

Verify Fubini's theorem for the following integrals of the function $f(x,y) = x\sqrt{x^2+y}$.
[Check your result: $I(1) = \frac{2}{15}\big(2^{5/2}-2\big)$.]

(a) $I(a) = \int_0^a \mathrm{d}x \int_0^1 \mathrm{d}y\ f(x,y)$, (b) $I(a) = \int_0^1 \mathrm{d}y \int_0^a \mathrm{d}x\ f(x,y)$.

Optional Problem 4: Fubini's theorem [2]
Points: [2](M)

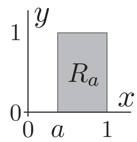
Verify Fubini's theorem for the following integrals of the function $f(x,y) = xy^2\sin(x^2+y^3)$.
[Check your result: $I(\sqrt{\pi/2}) = \frac{1}{3}$.]

(a) $I(a) = \int_0^a \mathrm{d}x \int_0^{\pi^{1/3}} \mathrm{d}y\ f(x,y)$, (b) $I(a) = \int_0^{\pi^{1/3}} \mathrm{d}y \int_0^a \mathrm{d}x\ f(x,y)$.

Optional Problem 5: Violation of Fubini's Theorem [Bonus]
Points: (a)[1](E,Bonus); (b)[0,5](E,Bonus); (M)[1](E,Bonus); (d)[0,5](M,Bonus).

Fubini's theorem holds only if the integrand is sufficiently well behaved that the integral of its *modulus* over the integration domain exists. Here we explore a counterexample.

(a) Integrate the function $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ over the rectangle $R_a = \{a \le x \le 1, 0 \le y \le 1\}$, with $0 < a \in \mathbb{R}$, using two different orders of integration:



$$I_\mathrm{A}(a) = \int_a^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, f(x,y)\,, \qquad\qquad I_\mathrm{B}(a) = \int_0^1 \mathrm{d}y \int_a^1 \mathrm{d}x\, f(x,y)\,.$$

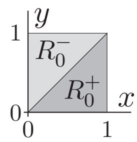
Verify that $I_\mathrm{A}(a) = I_\mathrm{B}(a)$. [Check your results: $I_\mathrm{A,B}(\sqrt{3}) = -\frac{\pi}{12}$.]

Hint: First show that $f(x,y) = \dfrac{\partial}{\partial y}\dfrac{y}{x^2+y^2} = -\dfrac{\partial}{\partial x}\dfrac{x}{x^2+y^2}$.

Set $a = 0$ for the remainder of this problem.

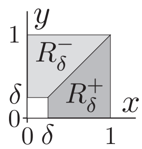
(b) Show that $I_\mathrm{A}(0) = -I_\mathrm{B}(0)$ if these integrals are recomputed, setting $a = 0$ from the outset. Which of the two, $I_\mathrm{A}(0)$ or $I_\mathrm{B}(0)$, agrees with the $a \to 0$ limit from part (a)?

(c) Show that the integral $I_\mathrm{C} = \int_{R_0} \mathrm{d}x\mathrm{d}y\, |f(x,y)|$ does not exist. To this end, split the integration domain $R_{a=0}$ into two parts, $R_0 = R_0^+ \cup R_0^-$, chosen such that $f \ge 0$ on R_0^+ and $f \le 0$ on R_0^- (see figure). Then $I_\mathrm{C} = \int_{R_0^+ \cup R_0^-} \mathrm{d}x\mathrm{d}y\, |f(x,y)| = I_0^+ - I_0^-$, with $I_0^\pm = \int_{R_0^\pm} \mathrm{d}x\, \mathrm{d}y\, f(x,y)$.
Compute the contributions $I_0^\pm$ separately and show that $I_0^+ = -I_0^- = \infty$.



As seen in (a) and (b), Fubini's theorem applies for $a > 0$, but not for $a = 0$, because then the integral over the *modulus* of the function does not exist, $I_\mathrm{C} = I_0^+ - I_0^- = \infty$, as seen in (c). This happens because for $a = 0$ the integration domain touches a point where f diverges — the origin: as $(x,y)^T$ approaches $(0,0)^T$, the integrand tends to $+\infty$ for $x > y$ or $-\infty$ for $x < y$. According to (c), the integrals over the positive or negative 'branches' of f diverge, $I_0^\pm = \pm\infty$. Hence the integral $I_0 = \int_{R_0} \mathrm{d}x\mathrm{d}y\, f(x,y)$ is *not defined*: it yields $\infty - \infty$ contributions, and the extent to which these cancel depends on the integration order, as seen in (b).

One may make sense of the integral I_0 by **regularizing** it, i.e. by modifying the integration domain to avoid the singularity. For example, consider the domain $R_\delta = R_0 \backslash S_\delta$, obtained from R_0 by removing an infinitesimal square adjacent to the origin, $S_\delta = \{0 \le x \le \delta, 0 \le y \le \delta\}$.



(d) Compute the integral $I_\delta = \int_{R_\delta} \mathrm{d}x\mathrm{d}y\, f(x,y)$ using the method of (c), splitting the integration domain as $R_\delta = R_\delta^+ \cup R_\delta^-$ (see figure). Discuss the limit $I_{\delta\to 0}$. Why is it well-defined?

Optional Problem 6: Violation of Fubini's Theorem [Bonus]

Points: (a)[1](E,Bonus); (b)[0,5](E,Bonus); (M)[1](E,Bonus); (d)[0,5](M,Bonus).

(a) Compute the integral of the function $f(x,y) = \frac{xy(x^2-y^2)}{(x^2+y^2)^3}$ over the rectangle $R_a = \{a \le x \le 1, 0 \le y \le 1\}$, with $0 < a \in \mathbb{R}$, using two different orders of integration:

$$I_\mathrm{A}(a) = \int_a^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, f(x,y)\,, \qquad\qquad I_\mathrm{B}(a) = \int_0^1 \mathrm{d}y \int_a^1 \mathrm{d}x\, f(x,y)\,.$$

Verify that $I_\mathrm{A}(a) = I_\mathrm{B}(a)$. [Check your results: $I_\mathrm{A,B}(\frac{1}{3}) = \frac{1}{10}$.]

Hint: First show that $f(x,y) = \frac{\partial}{\partial y}\frac{xy^2}{2(x^2+y^2)^2} = -\frac{\partial}{\partial x}\frac{x^2y}{2(x^2+y^2)^2}$.

(b) Show that $I_\mathrm{A}(0) = -I_\mathrm{B}(0)$ if these integrals are recomputed with $a = 0$ from the outset.

(c) Compute $I_\mathrm{C} = \int_{R_0} \mathrm{d}x\mathrm{d}y\,|f(x,y)|$ and explain why Fubini's theorem is violated in (b).

(d) Compute the regularized integral $I_\delta = \int_{R_\delta} \mathrm{d}x\mathrm{d}y f(x,y)$, where the integration domain $R_\delta = R_0 \backslash S_\delta$ is obtained from $R_{a=0}$ by removing an infinitesimal square adjacent to the origin, $S_\delta = \{0 \le x \le \delta, 0 \le y \le \delta\}$. Discuss the limit $I_{\delta\to 0}$. Why is it well-defined?

[Total Points for Optional Problems: 8]