Sheet 02: Vector Spaces, Euclidean Spaces

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 5, 7, 9, 8.
Videos exist for example problems 4 (L2.4.1), 9 (L3.3.7).

Optional Problem 1: Vector space of real functions [2]
Points: [2](M)

Let $F \equiv \{f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x)\}$ be the set of real functions. Show that $(F, \boldsymbol{+}, \boldsymbol{\cdot})$ is an $\mathbb{R}$-vector space, where the addition of functions, and their multiplication by scalars, are defined as follows:

$$\boldsymbol{+} : F \times F \to F \qquad (f, g) \mapsto f \boldsymbol{+} g, \qquad \text{with} \qquad f \boldsymbol{+} g : x \mapsto \big[f \boldsymbol{+} g\big](x) \equiv f(x) + g(x) \tag{1}$$

$$\boldsymbol{\cdot} : \mathbb{R} \times F \to F \qquad (\lambda, f) \mapsto \lambda \boldsymbol{\cdot} f, \qquad \text{with} \qquad \lambda \boldsymbol{\cdot} f : x \mapsto \big[\lambda \boldsymbol{\cdot} f\big](x) \equiv \lambda f(x) \tag{2}$$

Remark regarding notation: It is important to distinguish the 'name' of a function, f, from the 'function value', $f(x)$, which it returns when evaluated at the argument x. The sum of the functions f and g is a function named $f \boldsymbol{+} g$. Equation (1) states that its function value at x, denoted by $\big[f \boldsymbol{+} g\big](x)$, is by definition equal to $f(x) + g(x)$, the sum of the function values of f and g at x. (For emphasis, in this problem we use square bracket to indicate the function name; elsewhere we'll use round brackets for this.) The product of the number c and the function f yields a function named $c \boldsymbol{\cdot} f$. Eq. (2) states that its function value at x, denoted by $\big[c \boldsymbol{\cdot} f\big](x)$, is by definition equal to $cf(x)$, the product of c with the function value of f at x.

Optional Problem 2: Vector space of polynomials of degree at most n [3]
Points: (a)[1](E); (b)[1](E); (c)[1](E)

The vector space of all real functions is infinite-dimensional. However, if only functions of a prescribed form are considered, the corresponding vector space can be finite-dimensional. As an example, it is shown in this problem that the set of all polynomials of degree at most n form a vector space of dimension $n + 1$, isomorphic to $\mathbb{R}^{n+1}$.
[Remark on the notation: In the context of the present problem on polynomials, x^k means "x to the power of k", and a_k is "the coefficient of x^k". This is in contrast to the notation that we have adopted elsewhere when discussing vectors, where x^k stands for the k-th component of the vector $\mathbf{x} = \sum_k \mathbf{v}_k x^k$ with respect to a basis of vectors $\{\mathbf{v}_k\}$. Every notational convention has exceptions!]

Let $p_\mathbf{a}$ denote a polynomial in the variable $x \in \mathbb{R}$ of degree at most n:

$$p_\mathbf{a} : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto p_\mathbf{a}(x) \equiv a_0 x^0 + a_1 x^1 + \ldots a_n x^n.$$

$p_\mathbf{a}$ is uniquely specified by its $n+1$ real coefficients $a_0, a_1, \ldots, a_n$, which for notational brevity we arrange into a $(n+1)$-tuplet, $\mathbf{a} = (a_0, a_1, \ldots, a_n)^T \in \mathbb{R}^{n+1}$. Let $P_n = \{p_\mathbf{a} | \mathbf{a} \in \mathbb{R}^{n+1}\}$ denote the set of all such polynomials of degree n. The natural definitions for adding such polynomials, or multiplying them by a scalar $c \in \mathbb{R}$, are:

$$p_{\mathbf{a}} \boldsymbol{+} p_{\mathbf{b}} : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto p_{\mathbf{a}}(x) + p_{\mathbf{b}}(x)\,,$$
$$c \boldsymbol{\cdot} p_{\mathbf{a}} : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto c\, p_{\mathbf{a}}(x)\,,$$

where on the right side the usual addition and multiplication in $\mathbb{R}$ is used.

(a) Show that the above addition and scalar multiplication rules imply the following composition rules in P_n:

Addition of polynomials: $\quad \boldsymbol{+} : \quad P_n \times P_n \to P_n, \qquad (p_{\mathbf{a}}, p_{\mathbf{b}}) \mapsto p_{\mathbf{a}} \boldsymbol{+} p_{\mathbf{b}} \equiv p_{\mathbf{a}+\mathbf{b}}\,,$

Multiplication by a scalar: $\quad \boldsymbol{\cdot} : \quad \mathbb{R} \times P_n \to P_n, \qquad (c, p_{\mathbf{x}}) \mapsto c \boldsymbol{\cdot} p_{\mathbf{a}} \equiv p_{c\mathbf{a}}\,,$

where $\mathbf{a} + \mathbf{b}$ and $c\mathbf{a}$ denote the usual addition and scalar multiplication in $\mathbb{R}^{n+1}$.

(b) Show that $(P_n, \boldsymbol{+}, \boldsymbol{\cdot})$ is an $\mathbb{R}$-vector space, and that it is isomorphic to $\mathbb{R}^{n+1}$.

(c) Find a set $n + 1$ of polynomials, $\{p_{\mathbf{a}_0}, \dots, p_{\mathbf{a}_n}\} \subset P_n$, forming a basis for this vector space.

Optional Problem 3: Unconventional inner products on $\mathbb{R}^2$ [2]

Points: [2](M)

The defining properties of an inner product on $\mathbb{R}^n$ are of course satisfied not only by the 'standard' definition, $\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^{n} (x^i)^2$; there are infinitely many other bilinear forms that do so, too. The present problem illustrate this with a simple example. Show that the following map defines an inner product on the vector space $\mathbb{R}^2$:

$$\langle \cdot, \cdot \rangle \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \qquad \big(\mathbf{x}, \mathbf{y}\big) \mapsto x_1 y_1 + x_1 y_2 + x_2 y_1 + 3 x_2 y_2\,.$$

Optional Problem 4: Inner product and norm for the vector space of continuous functions [3]

Points: (a)[2](M); (b)[1](M)

This problem illustrates a particularly important example of an inner product: in the space of continuous functions, an inner product can be defined via integration.

Let V be the vector space of *continuous* real functions defined on a finite interval $I \subset \mathbb{R}$, $f : I \to \mathbb{R}$, with the usual composition rules of vector addition and scalar multiplication:

$$\forall f, g \in V : \qquad f + g : I \to \mathbb{R}, \qquad x \mapsto (f + g)(x) \equiv f(x) + g(x),$$
$$\forall f \in V, \lambda \in \mathbb{R} : \qquad \lambda \cdot f : I \to \mathbb{R}, \qquad x \mapsto \ (\lambda \cdot f)(x) \equiv \lambda\,(f(x))\,.$$

(a) Show that the following map defines an inner product on V:

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}, \qquad (f, g) \mapsto \langle f, g \rangle \equiv \int_I \mathrm{d}x\, f(x) g(x)\ .$$

(b) Now consider $I = [-1, 1]$. Compute $\langle f_1, f_2 \rangle$ for $f_1(x) \equiv \sin\left(\frac{x}{\pi}\right)$ and $f_2(x) \equiv \cos\left(\frac{x}{\pi}\right)$.

[Total Points for Optional Problems: 10]
