Sheet 01: Mathematical Foundations

Posted: Mo 17.10.22 Central Tutorial: Th 20.10.22 Due: Th 27.10.22, 14:00
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 9, 10, 4, 3.
Videos exist for example problems 9 (C2.3.1), 10 (C2.3.3).

Example Problem 1: Composition of maps [2]
Points: (a)[1](E); (b)[1](E).

Let $\mathbb{N}_0$ denote the set of all natural numbers including zero, and $\mathbb{Z}$ the set of all integers. Consider the following two maps:

$$A : \mathbb{Z} \to \mathbb{Z}, \qquad n \mapsto A(n) = n + 1,$$
$$B : \mathbb{Z} \to \mathbb{N}_0, \qquad n \mapsto B(n) = |n| \equiv n \cdot \mathrm{sign}(n).$$

(a) Find the composite map $C = B \circ A$, i.e. specify its domain, image and action on n.

(b) Which of the above maps A, B and C are surjective? Injective? Bijective?

Example Problem 2: The abelian group $\mathbb{Z}_2$ [3]
Points: (a)[2](E); (b)[1](E).

(a) Show that $\mathbb{Z}_2 \equiv (\{0, 1\}, +)$, where the addition operation $+$ is defined by the adjacent composition table, is an abelian group.

+	0	1
0	0	1
1	1	0

(b) Construct a group isomorphic to $\mathbb{Z}_2$, using two integers as group elements and standard multiplication of integers as group operation. Set up the corresponding composition table.

Example Problem 3: Permutation groups [4]
Points: (a)[3](E); (b)[0,5](E); (c)[0,5](E).

A map which reorders n labelled objects is called a **permutation** of these objects. For example, $1234 \overset{[4312]}{\longmapsto} 4312$ is a permutation of the four numbers in the string 1234, where we use $[4312]$ as shorthand for the map $1 \mapsto 4$, $2 \mapsto 3$, $3 \mapsto 1$ and $4 \mapsto 2$. Similarly, if the same permutation is applied to the string 2314, it yields $2314 \overset{[4312]}{\longmapsto} 3142$. (In general, $[P(1)...P(n)]$ denotes the map $j \mapsto P(j)$ which replaces j by $P(j)$, for $j = 1, ..., n$.) Two permutations performed in succession again yield a permutation. For example, acting on 1234 with $P = [4312]$ followed by $P' = [2413]$ yields $1234 \overset{[4312]}{\longmapsto} 4312 \overset{[2413]}{\longmapsto} 3124$, thus the resulting permutation is $P' \circ P = [3124]$.

The set of all possible permutations of n numbers, denoted by S_n, contains $n!$ elements. Viewing $P' \circ P$ (perform first P, then P') as a group operation,

$$\circ : S_n \times S_n \to S_n, \qquad (P', P) \mapsto P' \circ P\,,$$

we obtain a group, $(S_n, \circ)$, the **permutation group**, usually denoted simply by S_n.

(a) Complete the adjacent composition table for S_3, in which the entries $P' \circ P$ are arranged such that those with fixed P' sit in the same row, those with fixed P in the same column.

$P' \circ P$	[123]	[231]	[312]	[213]	[321]	[132]
[123]	[123]	[231]	[312]	[213]	[321]	[132]
[231]		[312]	[123]	[321]	[132]	[213]
[312]			[231]	[132]	[213]	[321]
[213]					[312]	[231]
[321]						[312]
[132]						

(b) Which element is the neutral element of S_3? How can we see from the multiplication table that every element has a unique inverse?

(c) Is S_3 an abelian group? Justify your answer.

Example Problem 4: Algebraic manipulations with complex numbers [4]
Points: (a-c)[0,5](E); (d)[0,5](M); (e)[0,5](E); (f)[0,5](E); (g)[1](M); (h)[1](M).

For $z = x + \mathrm{i}y \in \mathbb{C}$, bring each of the following expressions into standard form, i.e. write them as (real part) + i(imaginary part):

(a) $z + \bar{z}$, (b) $z - \bar{z}$, (c) $z \cdot \bar{z}$, (d) $\frac{z}{\bar{z}}$,

(e) $\frac{1}{z} + \frac{1}{\bar{z}}$, (f) $\frac{1}{z} - \frac{1}{\bar{z}}$, (g) $z^2 + z$, (h) z^3.

[Check your results for $x = 2$, $y = 1$: (a) 4, (b) $\mathrm{i}2$, (c) 5, (d) $\frac{3}{5} + \mathrm{i}\frac{4}{5}$, (e) $\frac{4}{5}$, (f) $-\mathrm{i}\frac{2}{5}$, (g) $5 + \mathrm{i}5$, (h) $2 + \mathrm{i}11$.]

Example Problem 5: Multiplication of complex numbers – geometrical interpretation [4]
Points: (a)[2](E); (b)[2](E)

(a) Consider the polar representation, $z_j = (\rho_j \cos\phi_j, \rho_j \sin\phi_j)$, of two complex numbers, z_1 and z_2, with $\phi_j \in [0, 2\pi)$. Show that multiplying them, $z_3 = z_1 z_2$, yields the relations $\rho_3 = \rho_1\rho_2$ and $\phi_3 = (\phi_1 + \phi_2)\mathrm{mod}(2\pi)$. [The $\mathrm{mod}(2\pi)$ is needed since we restricted polar angles to lie in the interval $[0, 2\pi)$.] To this end, the following trigonometric identities are useful:

$$\cos\phi_1 \cos\phi_2 - \sin\phi_1 \sin\phi_2 = \cos(\phi_1 + \phi_2),$$
$$\sin\phi_1 \cos\phi_2 + \cos\phi_1 \sin\phi_2 = \sin(\phi_1 + \phi_2).$$

(b) For $z_1 = \sqrt{3} + \mathrm{i}$, $z_2 = -2 + 2\sqrt{3}\mathrm{i}$, compute the product $z_3 = z_1 z_2$, as well as $z_4 = 1/z_1$ and $z_5 = \bar{z}_1$. Find the polar representation (with $\phi \in [0, 2\pi)$) of all five complex numbers and sketch them in the complex plane (in one diagram). Is your result for z_3 consistent with (a)?

Example Problem 6: Differentiation of trigonometric functions [1]
Points: (a)[0,5](E); (b)[0,5](E).

Show that the trigonometric functions

$$\tan x = \tfrac{\sin x}{\cos x}, \qquad \csc x = \tfrac{1}{\sin x}, \qquad \sec x = \tfrac{1}{\cos x}, \qquad \cot x = \tfrac{\cos x}{\sin x} = \tfrac{1}{\tan x},$$

satisfy the following identities:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\tan x = 1 + \tan^2 x = \sec^2 x$, (b) $\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -1 - \cot^2 x = -\csc^2 x$.

Example Problem 7: Differentiation of powers, exponentials, logarithms [2]
Points: [3](E).

Compute the first derivative of the following functions.
[Check your results against those in square brackets, where $[a, b]$ stands for $f'(a) = b$.]

(a) $f(x) = -\dfrac{1}{\sqrt{2x}}$ $\left[2, \frac{1}{8}\right]$

(b) $f(x) = \dfrac{x^{1/2}}{(x+1)^{1/2}}$ $\left[3, \frac{1}{16\sqrt{3}}\right]$

(c) $f(x) = \mathrm{e}^x(2x-3)$ $\left[1, \mathrm{e}\right]$

(d) $f(x) = 3^x$ $\left[-1, \frac{\ln 3}{3}\right]$

(e) $f(x) = x \ln x$ $\left[1, 1\right]$

(f) $f(x) = x\ln(9x^2)$ $\left[\frac{1}{3}, 2\right]$

Example Problem 8: Differentiation of inverse trigonometric functions [4]
Points: (a)[1](E); (b)[1](M); (c)[2](M).

Compute the following derivatives of inverse trigonometric functions, f^{-1}. For each case, make a qualitative scetch showing $f(x)$ and $f^{-1}(x)$. If f is non-monotonic, consider domains with positive or negative slope separately. [Check your results: $[a, b]$ stands for $(f^{-1})'(a) = b$.]

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x$ $[\frac{1}{3}, \frac{3}{\sqrt{8}}]$ (b) $\frac{\mathrm{d}}{\mathrm{d}x}\arccos x$ $[\frac{1}{2}, \frac{2}{\sqrt{3}}]$ (c) $\frac{\mathrm{d}}{\mathrm{d}x}\arctan x$ $[1, \frac{1}{2}]$

Hint: The identity $\sin^2 x + \cos^2 x = 1$ is useful for (a) and (b), $\sec^2 x = 1 + \tan^2 x$ for (c).

Example Problem 9: Integration by parts [6]
Points: [6](M)

Integrals of the form $I(z) = \int_{z_0}^{z} \mathrm{d}x\, u(x)v'(x)$ can be written as $I(z) = [u(x)v(x)]_{z_0}^{z} - \int_{z_0}^{z} \mathrm{d}x\, u'(x)v(x)$ using integration by parts. This is useful if $u'v$ can be integrated — either directly, or after further integrations by parts [see (b)], or after other manipulations [see (e,f)]. When doing such a calculation, it is advisable to clearly indicate the factors u, v', v and u'. Always check that the derivative $I'(z) = \mathrm{d}I/\mathrm{d}z$ of the result reproduces the integrand! If a single integration by parts suffices to calculate $I(z)$, its derivative exhibits the cancellation pattern $I' = u'v + uv' - u'v = uv'$ [see (a,c,d)]; otherwise, more involved cancellations occur [see (b,e,f)].
Integrate the following integrals by parts. [Check your results against those in square brackets, where $[a, b]$ stands for $I(a) = b$.]

(a) $I(z) = \displaystyle\int_0^z \mathrm{d}x\; x\, \mathrm{e}^{2x}$ $\left[\frac{1}{2}, \frac{1}{4}\right]$

(b) $I(z) = \displaystyle\int_0^z \mathrm{d}x\; x^2\, \mathrm{e}^{2x}$ $\left[\frac{1}{2}, \frac{e}{8} - \frac{1}{4}\right]$

(c) $I(z) = \displaystyle\int_0^z \mathrm{d}x\; \ln x$ $\left[1, -1\right]$

(d) $I(z) = \displaystyle\int_0^z \mathrm{d}x\; \ln x \frac{1}{\sqrt{x}}$ $\left[1, -4\right]$

(e) $I(z) = \displaystyle\int_0^z \mathrm{d}x\; \sin^2 x$ $\left[\pi, \frac{\pi}{2}\right]$

(f) $I(z) = \displaystyle\int_0^z \mathrm{d}x\; \sin^4 x$ $\left[\pi, \frac{3\pi}{8}\right]$

Example Problem 10: Integration by substitution [4]
Points: [4](M)

Integrals of the form $I(z) = \int_{z_0}^{z} \mathrm{d}x\, y'(x) f(y(x))$ can be written as $I(z) = \int_{y(z_0)}^{y(z)} \mathrm{d}y f(y)$ by using the substitution $y = y(x)$, $\mathrm{d}y = y'(x)\mathrm{d}x$. When doing such integrals, it is advisable to explicitly write down $y(x)$ and $\mathrm{d}y$, to ensure that you correctly identify the prefactor of $f(y)$. Always check that the derivative $I'(z) = \mathrm{d}I/\mathrm{d}z$ of the result reproduces the integrand! You'll notice that the factor $y'(z)$ emerges via the chain rule for differentiating composite functions.

Calculate the following integrals by substitution. [Check your results against those in square brackets, where $[a, b]$ stands for $I(a) = b$.]

(a) $I(z) = \int_0^z \mathrm{d}x\, x\cos(x^2 + \pi)$ $\quad \left[\sqrt{\frac{\pi}{2}}, -\frac{1}{2}\right]$

(b) $I(z) = \int_0^z \mathrm{d}x \sin^3 x \cos x$ $\quad \left[\frac{\pi}{4}, \frac{1}{16}\right]$

(c) $I(z) = \int_0^z \mathrm{d}x \sin^3 x$ $\quad \left[\frac{\pi}{3}, \frac{5}{24}\right]$

(d) $I(z) = \int_0^z \mathrm{d}x \cosh^3 x$ $\quad \left[\ln 2, \frac{57}{64}\right]$

(e) $I(z) = \int_0^z \mathrm{d}x \frac{\sqrt{1 + \ln(x+1)}}{x+1}$ $\quad \left[\mathrm{e}^3 - 1, \frac{14}{3}\right]$

(f) $I(z) = \int_0^z \mathrm{d}x\, x^3 \mathrm{e}^{-x^4}$ $\quad \left[\sqrt[4]{\ln 2}, \frac{1}{8}\right]$

[Total Points for Example Problems: 34]

Homework Problem 1: Composition of maps [2]
Points: (a)[0,5](E); (b)[0,5](E); (c)[0,5](E); (d)[0,5](E).

(a) Consider the set $S = \{-2, -1, 0, 1, 2\}$. Find its image, $T = A(S)$, under the map $n \mapsto A(n) = n^2$. Is the map $A : S \to T$ surjective? Injective? Bijective?

(b) Find the image, $U = B(T)$, of the set T from part (a) under the map $n \mapsto B(n) = \sqrt{n}$.

(c) Find the composite map $C = B \circ A$.

(d) Which of the above maps A, B and C are surjective? Injective? Bijective?

Homework Problem 2: The groups of addition modulo 5 and rotations by multiples of 72 deg [3]
Points: (a)[1](E); (b)[1](E); (c)[0,5](E); (d)[0,5](E).

(a) Consider the set $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, endowed with the group operation

$$\boldsymbol{+} : \mathbb{Z}_5 \times \mathbb{Z}_5 \to \mathbb{Z}_5, \qquad (p, p') \mapsto p \boldsymbol{+} p' \equiv (p + p') \bmod 5.$$

Set up the composition table for the group $(\mathbb{Z}_5, \boldsymbol{+})$. Which element is the neutral element? For a given $n \in \mathbb{Z}$, which element is the inverse of n?

(b) Let $r(\phi)$ denote a rotation by ϕ degrees about a fixed axis, with $r(\phi + 360) = r(\phi)$. Consider the set of rotations by multiples of 72 deg,

$$\mathcal{R}_{72} = \{r(0), r(72), r(144), r(216), r(288)\},$$

and the group $(\mathcal{R}_{72}, \bullet)$, where the group operation $\bullet$ involves two rotations in succession:

$$\bullet : \mathcal{R}_{72} \times \mathcal{R}_{72} \to \mathcal{R}_{72}, \qquad (r(\phi), r(\phi')) \mapsto r(\phi) \bullet r(\phi') \equiv r(\phi + \phi').$$

Set up the multiplication table for this group. Which element is the neutral element? Which element is the inverse of $r(\phi)$?

(c) Explain why the groups $(\mathbb{Z}_5, +)$ and $(\mathcal{R}_{72}, \bullet)$ are isomorphic.

(d) Let $(\mathbb{Z}_n, +)$ denote the group of integer addition modulo n of the elements of the set $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. Which group of discrete rotations is isomorphic to this group?

Homework Problem 3: Decomposing permutations into sequences of pair permutations [2]

Consider the permutation group S_n. Any permutation can be decomposed into a sequence of **pair permutations**, i.e. permutations which exchange just two objects, leaving the others unchanged. Examples:

$$\begin{aligned}
123 &\overset{[321]}{\longmapsto} 321 \overset{[132]}{\longmapsto} 231 &&\Rightarrow \quad [231] = [132] \circ [321]. \\
1234 &\overset{[2134]}{\longmapsto} 2134 \overset{[3214]}{\longmapsto} 2314 &&\Rightarrow \ [2314] = [3214] \circ [2134], \\
1234 &\overset{[3214]}{\longmapsto} 3214 \overset{[1324]}{\longmapsto} 2314 &&\Rightarrow \ [2314] = [1324] \circ [3214], \\
1234 &\overset{[4231]}{\longmapsto} 4231 \overset{[1432]}{\longmapsto} 2431 \overset{[1243]}{\longmapsto} 2341 \overset{[4231]}{\longmapsto} 2314 &&\Rightarrow \ [2314] = [4231] \circ [1243] \circ [1432] \circ [4231].
\end{aligned}$$

The last three lines illustrate that a given permutation can be pair-decomposed in several ways, and that these may or may not involve different numbers of pair exchanges. However, one may convince oneselve (try it!) that all pair decompositions of a given permutation have the same **parity**, i.e. the number of exchanges is either always **even** or always **odd**.

To find a 'minimal' (shortest possible) pair decomposition of a given permutation, say $[2413]$, we may start from the naturally-ordered string 1234 and rearrange it to its desired form, 2413, one pair permutation at a time, bringing the 2 to the first slot, then the 4 to the second slot, etc. This yields $1234 \overset{[2134]}{\longmapsto} 2134 \overset{[4231]}{\longmapsto} 2431 \overset{[3214]}{\longmapsto} 2413$, hence $[2413] = [3214] \circ [4231] \circ [2134]$.

Find a minimal pair decomposition and the parity of each of the following permutations:

(a) $[132]$, (b) $[231]$, (c) $[3412]$, (d) $[3421]$, (e) $[15234]$, (f) $[31542]$.

Homework Problem 4: Algebraic manipulations with complex numbers [3]
Points: (a)[1](E); (b)[1](M); (c)[1](E).

For $z = x + \mathrm{i}y \in \mathbb{C}$, bring each of the following expressions into standard form:

(a) $(z + \mathrm{i})^2$, (b) $\dfrac{z}{z+1}$, (c) $\dfrac{\bar{z}}{z - \mathrm{i}}$.

[Check your results for $x = 1$, $y = 2$: (a) $-8 + \mathrm{i}6$, (b) $\frac{3}{4} + \mathrm{i}\frac{1}{4}$, (c) $-\frac{1}{2} - \mathrm{i}\frac{3}{2}$.]

Homework Problem 5: Multiplication of complex numbers – geometrical interpretation [2]
Points: [2](E)

For $z_1 = \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{8}}\mathrm{i}$, $z_2 = \sqrt{3} - \mathrm{i}$, compute the product $z_3 = z_1 z_2$, as well as $z_4 = 1/z_1$ and $z_5 = \bar{z}_1$. Find the polar representation (with $\phi \in [0, 2\pi)$) of all five complex numbers and sketch them in the complex plane (in one diagram).

Homework Problem 6: Differentiation of hyperbolic functions [2]
Points: (a)[0,5](E); (b,c)[0,5](E); (d)[0,5](E); (e)[0,5](E).

Show that the hyperbolic functions

$$\sinh x = \tfrac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}), \qquad \cosh x = \tfrac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}), \qquad \tanh x = \tfrac{\sinh x}{\cosh x},$$
$$\operatorname{csch} x = \tfrac{1}{\sinh x}, \qquad \operatorname{sech} x = \tfrac{1}{\cosh x}, \qquad \coth x = \tfrac{\cosh x}{\sinh x} = \tfrac{1}{\tanh x},$$

satisfy the following identities:

(a) $\cosh^2 x - \sinh^2 x = 1$,

(b) $\frac{\mathrm{d}}{\mathrm{d}x} \sinh x = \cosh x$,

(c) $\frac{\mathrm{d}}{\mathrm{d}x} \cosh x = \sinh x$.

(d) $\frac{\mathrm{d}}{\mathrm{d}x} \tanh x = 1 - \tanh^2 x = \operatorname{sech}^2 x$,

(e) $\frac{\mathrm{d}}{\mathrm{d}x} \coth x = 1 - \coth^2 x = -\operatorname{csch}^2 x$.

Homework Problem 7: Differentiation of powers, exponentials, logarithms [2]
Points: [2](E) (Solve any 4 subproblems; beyond that: 0.25 bonus per subproblem.)

Compute the first derivative of the following functions.
[Check your results against those in square brackets, where $[a, b]$ stands for $f'(a) = b$.]

(a) $f(x) = \sqrt[3]{x^2}$ $\quad [8, \frac{1}{3}]$

(b) $f(x) = \dfrac{x}{(x^2+1)^{1/2}}$ $\quad \left[1, \frac{1}{\sqrt{8}}\right]$

(c) $f(x) = -\mathrm{e}^{(1-x^2)}$ $\quad [1, 2]$

(d) $f(x) = 2^{x^2}$ $\quad [1, 4\ln 2]$

(e) $f(x) = 2\dfrac{\sqrt{\ln x}}{x}$ $\quad [\mathrm{e}, -\frac{1}{\mathrm{e}^2}]$

(f) $f(x) = \ln\sqrt{x^2+1}$ $\quad [1, \frac{1}{2}]$

Homework Problem 8: Differentiation of inverse hyperbolic functions [2]
Points: [2](M) (Solve subproblem, (b); beyond that: 0.5 bonus points per subproblem.)

Compute the following derivatives of inverse hyperbolic functions, f^{-1}. For each case, make a qualitative sketch showing $f(x)$ and $f^{-1}(x)$. If f is non-monotonic, consider domains with positive or negative slope separately. [Check your results: $[a, b]$ stands for $(f^{-1})'(a) = b$.]

(a) $\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{arcsinh} x$ $\quad [2, \frac{1}{\sqrt{5}}]$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{arccosh} x$ $\quad [2, \frac{1}{\sqrt{3}}]$

(c) $\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{arctanh} x$ $\quad [\frac{1}{2}, \frac{4}{3}]$

Hint: The identity $\cosh^2 x = 1 + \sinh^2 x$ is useful for (a) and (b), $\operatorname{sech}^2 x = 1 - \tanh^2 x$ for (c).

Homework Problem 9: Integration by parts [4]
Points: [4](M) (Solve any 4 subproblems; beyond that: 0.5 bonus per subproblem.)

Integrate the following integrals by parts. [Check your results against those in square brackets, where $[a, b]$ stands for $I(a) = b$.]

(a) $I(z) = \displaystyle\int_0^z \mathrm{d}x\ x\sin(2x)$ $\quad [\frac{\pi}{2}, \frac{\pi}{4}]$

(b) $I(z) = \displaystyle\int_0^z \mathrm{d}x\ x^2\cos(2x)$ $\quad [\frac{\pi}{2}, -\frac{\pi}{4}]$

(c) $I(z) = \int_0^z \mathrm{d}x \ (\ln x) \ x$ $\left[1, -\frac{1}{4}\right]$

(d) $I(z) \overset{[n>-1]}{=} \int_0^z \mathrm{d}x \ (\ln x) \ x^n$ $\left[1, \frac{-1}{(n+1)^2}\right]$

(e) $I(z) = \int_0^z \mathrm{d}x \ \cos^2 x$ $\left[\pi, \frac{\pi}{2}\right]$

(f) $I(z) = \int_0^z \mathrm{d}x \ \cos^4 x$ $\left[\pi, \frac{3}{8}\pi\right]$

Homework Problem 10: Integration by substitution [3]

Points: [3](M) (Solve any 3 subproblems; beyond that: 0.5 bonus per subproblem.)

Calculate the following integrals by substitution. [Check your results versus those in square brackets, where $[a, b]$ stands for $I(a) = b$.]

(a) $I(z) = \int_0^z \mathrm{d}x \, x^2 \ \sqrt{x^3 + 1}$ $\left[2, \frac{52}{9}\right]$

(b) $I(z) = \int_0^z \mathrm{d}x \sin x \ \mathrm{e}^{\cos x}$ $\left[\frac{\pi}{3}, \mathrm{e} - \sqrt{\mathrm{e}}\right]$

(c) $I(z) = \int_0^z \mathrm{d}x \cos^3 x$ $\left[\frac{\pi}{4}, \frac{5}{6\sqrt{2}}\right]$

(d) $I(z) = \int_0^z \mathrm{d}x \sinh^3 x$ $\left[\ln 3, \frac{44}{81}\right]$

(e) $I(z) = \int_0^z \mathrm{d}x \frac{\sin\sqrt{\pi x}}{\sqrt{x}}$ $\left[\frac{\pi}{9}, \frac{1}{\sqrt{\pi}}\right]$

(f) $I(z) = \int_0^z \mathrm{d}x \sqrt{x} \ \mathrm{e}^{\sqrt{x^3}}$ $\left[(\ln 4)^{2/3}, 2\right]$

[Total Points for Homework Problems: 25]
