Sheet 01: Mathematical Foundations

(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 9, 10, 4, 3.
Videos exist for example problems 9 (C2.3.1), 10 (C2.3.3).

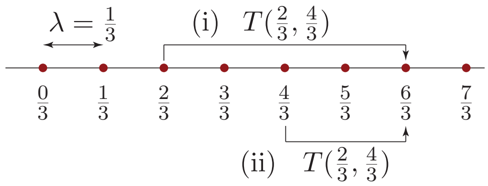
Optional Problem 1: Group of discrete translations in one dimension [4]

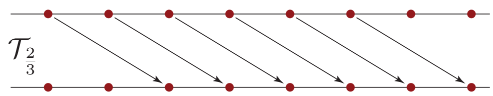
Points: (a)[2](E); (b)[2](M)

In this problem we show that discrete translations on an infinite, one-dimensional lattice form a group. Let us denote the lattice constant, i.e. the fixed distance between neighboring lattice points, by $\lambda \in \mathbb{R}^+$, a positive, real number. The lattice $\mathbb{G}$ consists of the set of all integer multiples of λ, i.e. $\mathbb{G} \equiv \lambda\mathbb{Z} \equiv \{x \in \mathbb{R} | \exists n \in \mathbb{Z} : x = \lambda \cdot n\}$, where $\cdot$ is the usual multiplication rule in $\mathbb{R}$. Note that for any given $x \in \mathbb{G}$, n is uniquely determined. On this lattice we define 'translation' by the group operation

$$T: \quad \mathbb{G} \times \mathbb{G} \to \mathbb{G}, \quad (x, y) \mapsto T(x, y) \equiv x + y,$$

where $+$ denotes the usual addition of real numbers. Since this operation is symmetric, it can be visualized in two equivalent ways: $T(x, y)$ describes (i) a 'shift' or a 'translation' of lattice point x by the distance y, or (ii) a translation of lattice point y by the distance x. [Figure (a), where $\lambda = \frac{1}{3}$, shows both visualizations of $T(\frac{2}{3}, \frac{4}{3})$.]

(a) 

(b) 

(a) Show that $(\mathbb{G}, T)$ forms an abelian group.

(b) For a given $y \in \mathbb{G}$ we now define, in accordance with visualization (i), a 'translation' of the lattice by y, i.e. each lattice point x is 'shifted' by y:

$$\mathcal{T}_y: \quad \mathbb{G} \to \mathbb{G}, \quad x \mapsto \mathcal{T}_y(x) \equiv T(x, y).$$

[Figure (b), where $\lambda = \frac{1}{3}$, shows $\mathcal{T}_{\frac{2}{3}}$.] Now consider the set of all such translations, $\mathbb{T} \equiv \{\mathcal{T}_y, y \in \mathbb{G}\}$. Show that $(\mathbb{T}, \boldsymbol{+})$ forms an abelian group, where $\boldsymbol{+}$ is defined as

$$\boldsymbol{+}: \quad \mathbb{T} \times \mathbb{T} \to \mathbb{T}, \quad (\mathcal{T}_x, \mathcal{T}_y) \mapsto \mathcal{T}_x \boldsymbol{+} \mathcal{T}_y \equiv \mathcal{T}_{T(x,y)}.$$

Remark: the set $\mathbb{T}$ underlying this group consists of maps (namely translations), illustrating that the set underlying a group need not be 'simple'.

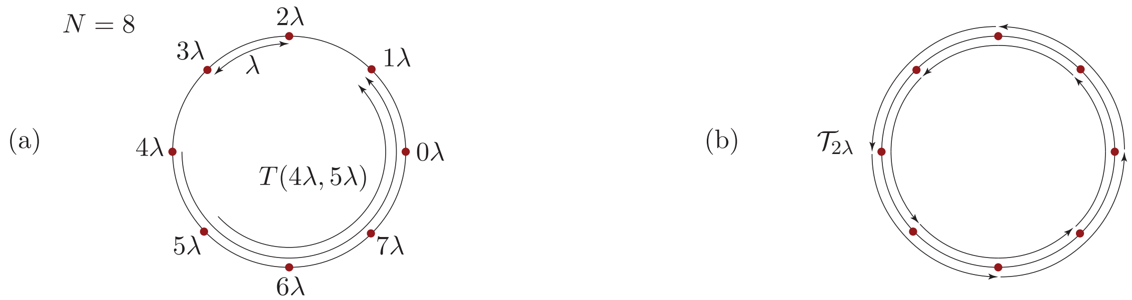
Optional Problem 2: Group of discrete translations on a ring [4]

Points: (a)[2](M); (b)[2](M)

In this problem we show that discrete translations on a finite, one-dimensional lattice with periodic boundary conditions form a group. Consider a ring with radius $0 < R \in \mathbb{R}$ and lattice constant $\lambda = 2\pi R/N$ with $N \in \mathbb{N}$, thus $\mathbb{G} \equiv \lambda(\mathbb{Z} \bmod N) \equiv \{x \in \mathbb{R} | \exists n \in \{0, 1, \ldots, N-1\} : x = \lambda \cdot n\}$, where $\cdot$ is the usual multiplication rule in $\mathbb{R}$. Note that for any given $x \in \mathbb{G}$, n is uniquely determined. The ring forms a 'periodic' structure: when counting its sites, 0λ and $N\lambda$ describe the same lattice site, the same is true for 1λ and $(1+N)\lambda$, for 2λ and $(2+N)\lambda$, etc. On this lattice we define a group operation, corresponding to a 'translation', using addition modulo N:

$$T: \quad \mathbb{G} \times \mathbb{G} \to \mathbb{G}, \quad (x, y) = (\lambda \cdot n_x, \lambda \cdot n_y) \mapsto T(x, y) \equiv \lambda \cdot ((n_x + n_y) \bmod N).$$

Here $+$ is the usual addition of integers, and $n \bmod N$ (spoken as 'n mod N') is defined as the integer remainder after division of n by N (e.g. $9 \bmod 8 = 1$). [For $N = 8$, figure (a) shows two visualizations of the translation $T(4\lambda, 5\lambda)$: as a 'shift' of the lattice site 4λ by the distance 5λ along the ring, or of the site 5λ by the distance 4λ.]



(a) Show that $(\mathbb{G}, T)$ forms an abelian group.

(b) For a given $y \in \mathbb{G}$ we now define a 'translation' of the lattice by y,

$$\mathcal{T}_y: \quad \mathbb{G} \to \mathbb{G}, \quad x \mapsto \mathcal{T}_y(x) \equiv T(x, y)$$

i.e. each site x is 'shifted' by y along the ring. [For $N = 8$, figure (b) shows the translation $\mathcal{T}_{2\lambda}$]. Now consider the set of all such translations, $\mathbb{T} \equiv \{\mathcal{T}_y, y \in \mathbb{G}\}$. Show that $(\mathbb{T}, \boldsymbol{+})$ forms an abelian group, where the group operation $\boldsymbol{+}$ is defined as

$$\boldsymbol{+}: \quad \mathbb{T} \times \mathbb{T} \to \mathbb{T}, \quad (\mathcal{T}_x, \mathcal{T}_y) \mapsto \mathcal{T}_x \boldsymbol{+} \mathcal{T}_y \equiv \mathcal{T}_{T(x,y)}.$$

Optional Problem 3: L'Hôpital's rule [4]

Points: (a)[0,5](E); (b)[0,5](E); (c)[1](M); (d)[1](M); (e)[1](M)

Consider the following question: what is the limiting value of the ratio, $\lim_{x \to x_0} \frac{f(x)}{g(x)}$, if the functions f and g both vanish at the point x_0? The naive answer, $\frac{f(x_0)}{g(x_0)} \stackrel{?}{=} \frac{0}{0}$, is ill-defined. However, if both functions have a finite slope at x_0, we may use a linear approximation for both, $f(x_0 + \delta) \simeq 0 + \delta f'(x_0)$ and $g(x_0 + \delta) \simeq 0 + \delta g'(x_0)$, to obtain $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$. This result is a special case of L'Hôpital's rule.

The general formulation of L'Hôpital's rule is: If either $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$ or $\lim_{x \to x_0} |f(x)| = \lim_{x \to x_0} |g(x)| = \infty$, and the limit $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}. \tag{1}$$

The proof of this general statement is non-trivial, but is a standard topic in calculus textbooks.

Use L'Hôpital's rule to evaluate the following limits as functions of the real number a: [Check your results against those in square brackets, where $[a, b]$ means that the limit $L(a) = b$.]

(a) $\lim_{x\to 1} \dfrac{x^2 + (a-1)x - a}{x^2 + 2x - 3}$ $[3, 1]$ (b) $\lim_{x\to 0} \dfrac{\sin(ax)}{x + ax^2}$ $[2, 2]$

If not only f and g but also f' and g' all vanish at x_0, the limit on the r.h.s. of L'Hôpital's rule may be evaluated by applying the rule a second time (or $n+1$ times, if the derivatives up to $f^{(n)}$ and $g^{(n)}$ all vanish at x_0). Use this procedure to evaluate the following limits:

(c) $\lim_{x\to 0} \dfrac{1 - \cos(ax)}{\sin^2 x}$, $[4, 8]$ (d) $\lim_{x\to 0} \dfrac{x^3}{\sin(ax) - ax}$. $[2, -\frac{3}{4}]$

(e) Use L'Hôpital's rule to show that $\lim_{x\to 0}(x \ln x) = 0$ (with $x > 0$). This result implies that for $x \to 0$, 'x decreases more quickly than $\ln(x)$ diverges', i.e. 'linear beats log'.

Optional Problem 4: L'Hôpital's rule [4]

Points: (a)[0,5](E); (b)[0,5](E); (c)[1](M); (d)[1](M); (e)[1](M)

Use L'Hôpital's rule (possibly multiple times) to evaluate the following limits as functions of the real number a: [Check your results: $[a, b]$ means that the limit $L(a) = b$.]

(a) $\lim_{x\to a} \dfrac{x^2 + (2-a)x - 2a}{x^2 - (a+1)x + a}$ $[2, 4]$ (b) $\lim_{x\to 0} \dfrac{\sinh(x)}{\tanh(ax)}$ $[2, \frac{1}{2}]$

(c) $\lim_{x\to 0} \dfrac{e^{x^2} - 1}{(e^{ax} - 1)^2}$ $[2, \frac{1}{2}]$ (d) $\lim_{x\to 0} \dfrac{\cosh(ax) + \cos(ax) - 2}{x^4}$ $[2, \frac{4}{3}]$

(e) Use L'Hôpital's rule to show that for $\alpha \in \mathbb{R}$ and $0 < \beta \in \mathbb{R}$ we have

$$\lim_{x\to 0}(x^\beta \ln^\alpha x) = 0 \quad \text{(with } x > 0\text{)},$$

i.e. 'any positive power law beats any power of log'.

[Total Points for Optional Problems: 16]
