Trigonometrische Funktionen

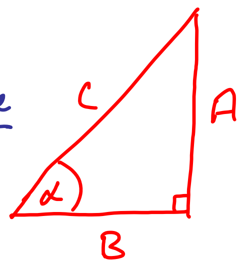
Folgende Seiten sind ein Auszug aus dem Mathe-Crash-Kurs 2013, gehalten von Jan von Delft
Links zu allen Folien und Videos dieser Veranstaltung:
Moodle-T0-21/22 -> Info -> Mathevorkurs (ganz unten!)

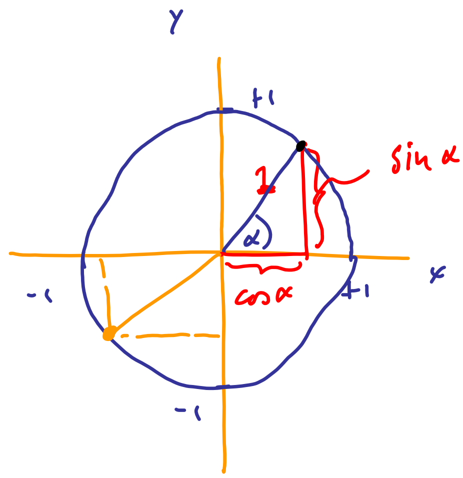
schwarz: wird für T0 als Grundwissen (aus der Schule bekannt) vorausgesetzt
braun: wird in T0- Vorlesung und/oder Übungen behandelt

1.4.4 Trigonometrische Funktionen

In rechtwinkligem Dreieck:

$\sin\alpha \equiv \frac{A}{C} = \frac{\text{Gegenkathete}}{\text{Hypotenuse}}$





$\cos\alpha \equiv \frac{B}{C} = \frac{\text{Ankathete}}{\text{Hypothenuse}}$

Pythagoras: $A^2 + B^2 = C^2 \Rightarrow \frac{A^2}{C^2} + \frac{B^2}{C^2} = 1$

$\sin^2\alpha + \cos^2\alpha = 1$

Def: Sinusfunktion $\sin : \mathbb{R} \longrightarrow [-1,1]$ $\mathbb{R}$

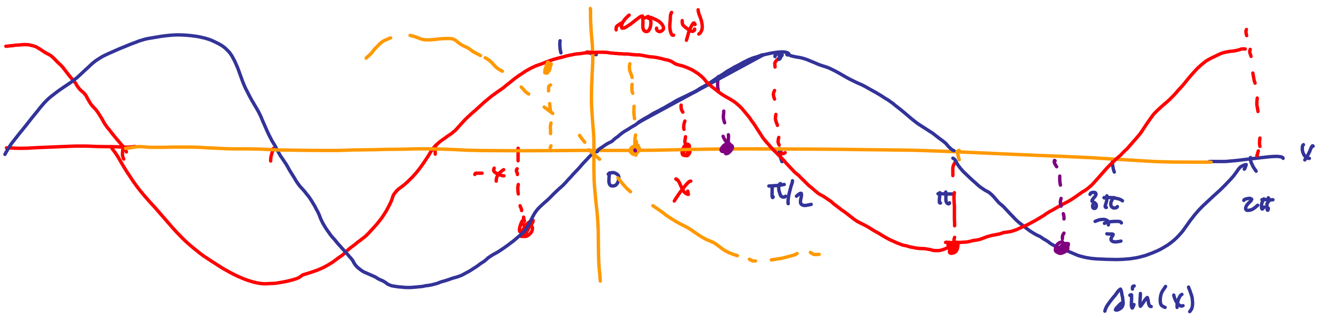
$x \longmapsto \sin x$, mit $\sin(x) = \sin(x + n2\pi)$

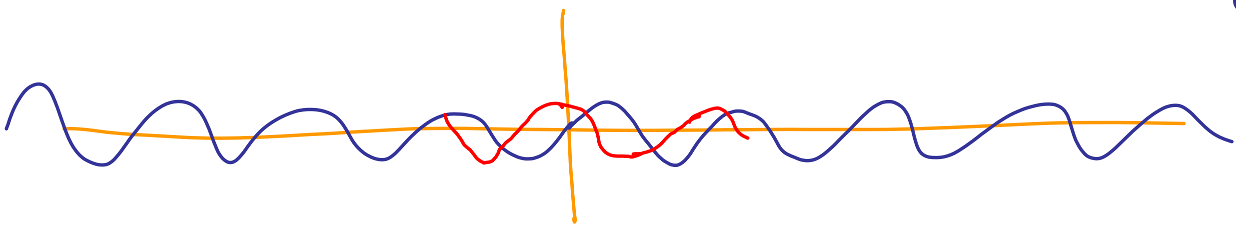
$n \in \mathbb{Z}$ "periodische Fortsetzung"

Cosinusfunktion $\cos : \mathbb{R} \longrightarrow [-1,1]$ $\mathbb{R}$

$x \longmapsto \cos x$, mit $\cos(x) = \cos(x + n2\pi)$

$360^\circ = 2\pi$





Sin & cos sind periodische Fktn, mit Periode 2π.

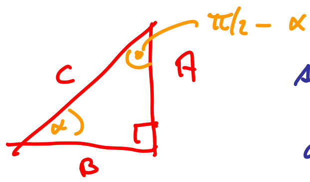
Allg: "periodische Fkt" hat Periode L falls: $f(x+L) = f(x)$ $\forall x$.

$\sin(-x) = -\sin(x)$ ("punktsymmetrisch zum Ursprung")

$\cos(-x) = \cos x$ ("Achsensymmetrisch zur y-Achse")

$\sin(\pi/2 - x) = \cos x = B/C$

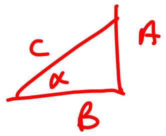
$\cos(\pi/2 - x) = \sin x = A/C$



$\sin(x+\pi) = -\sin x$

$\cos(x+\pi) = -\cos x$

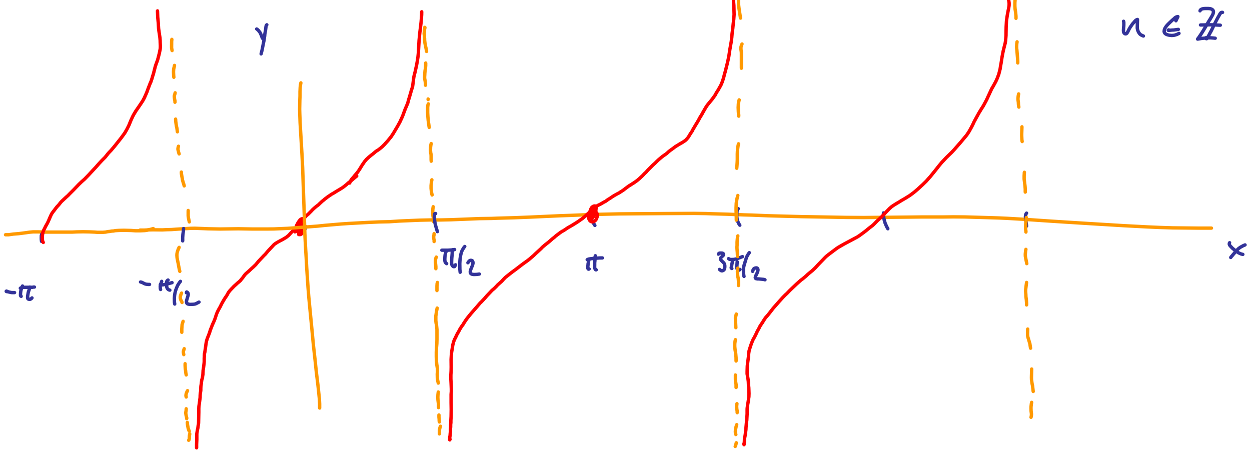
Tangens:



$$\tan\alpha = \frac{A}{B} = \frac{(A/C)}{(B/C)} = \frac{\sin\alpha}{\cos\alpha}$$

$$f : \mathbb{R} \setminus \{\overbrace{\text{Nullstellen des Cosinus}}^{\pi/2 + m\pi,\ m \in \mathbb{Z}}\} \longrightarrow \mathbb{R}$$

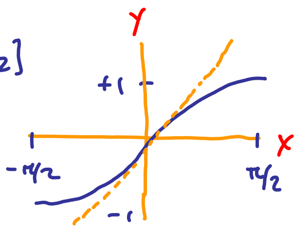
$$x \longmapsto \tan(x) = \frac{\sin x}{\cos x}, \quad \text{mit } \tan(x) = \tan(x + \pi n),\ n \in \mathbb{Z}$$



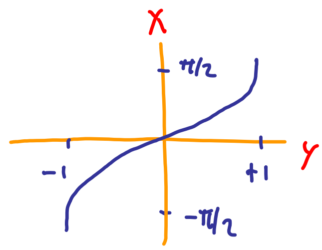
Umkehrfunktionen v. Trig. Funktionen:

Sin, Cos, Tan, sind im gesammten Def-Bereich nicht umkehrbar, wohl aber in Teilbereichen, wo sie monoton sind.

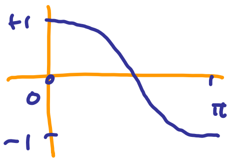
sin in $[-\pi/2, \pi/2]$



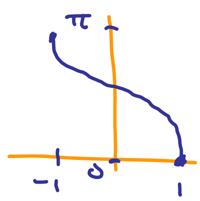
arsin "Arcsinus"



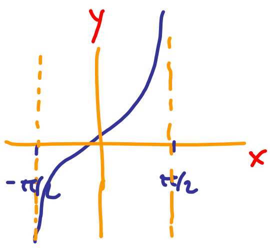
cos in $[0, \pi]$



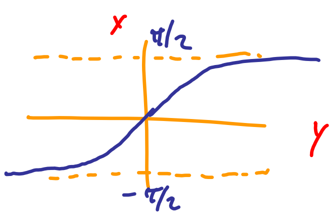
arcos "Arcosinus"



tan in $[-\pi/2, \pi/2]$



arctan "Arctangens"



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$$\arcsin\left[\sin(x)\right] = x \qquad \text{Bsp: } x = \pi/2, \ \sin x = 1$$

$$\arcsin 1 = \pi/2 \ ✓$$

$$\arccos\left[\cos(x)\right] = x$$

Additionstheoreme:

$$\sin(A+B) = \sin A \cdot \cos B + \sin B \cos A$$

$$\cos(A+B) = \cos A \cdot \cos B - \sin A \sin B$$

Beweis: TO - Vorlesung

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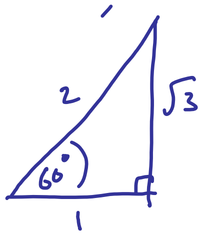
Spezielle Winkel: $\alpha = 45°$, $\alpha = \pi/4$

$$\Rightarrow \sin \pi/4 = \frac{1}{\sqrt{2}}, \quad \cos \pi/4 = \frac{1}{\sqrt{2}}$$

$\alpha = 60°$, $\alpha = \pi/2 \cdot 2/3 = \pi/3$



$$\sin \pi/3 = \frac{\sqrt{3}}{2}$$

$$\cos \pi/3 = \frac{1}{2}$$

$$\sin(x + \pi/3) = \sin x \cos \pi/3 + \sin \pi/3 \cos x$$

$$= \frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x$$

3.2 Ableitung

$f(x)$ ist bei x_0 "differenzierbar" falls $\underbrace{\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}}_{\equiv \text{ "Ableitung v. f. bei } x_0\text{"}}$ existiert.

Notationen:

$$f'(x_0) \equiv \frac{df}{dx}(x_0) = \frac{d}{dx} f(x)\Big|_{x=x_0} \equiv \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = d_x f(x)\Big|_{x_0}$$

"f-Strich" "df nach dx" bei x_0 "d nach dx von f(x)" bei x_0 "d-x von f"

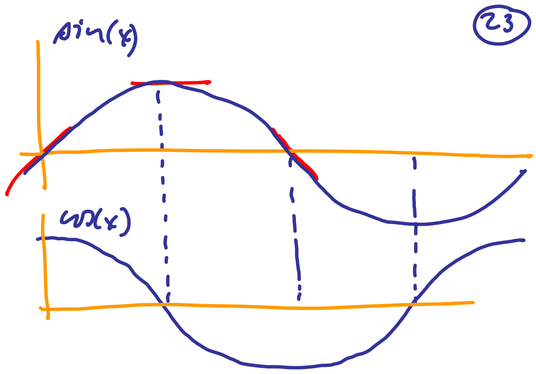
$f'(x_0)$ gibt Steigung an von der Tangente an den Graph G_f im Punkt $(x_0, f(x_0))$.

Satz: Jede differenzierbare Fkt. ist stetig.

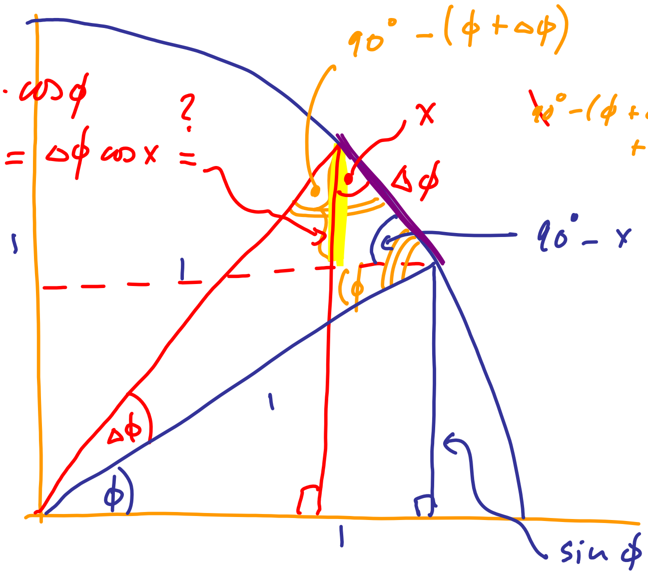
Trigonometrische Funktionen:

$u(x) = \sin(x)$, $u'(x) = \cos x$

$v(x) = \cos(x)$, $v'(x) = -\sin x$



$\Delta\phi \cdot \cos\phi$
$= \Delta\phi \cos x \overset{?}{=}$



$$90^\circ - (\phi+\Delta\phi) + x = \qquad = \qquad = 90^\circ - x + \phi$$

$$2x = 2\phi - \Delta\phi$$

$$x = \phi - \cancel{\Delta\phi/2}$$

$$\sin'\phi = \lim_{\Delta\phi\to 0} \frac{\sin(\phi+\Delta\phi) - \sin\phi}{\Delta\phi}$$

$$= \lim_{\Delta\phi\to 0} \frac{\cancel{\Delta\phi}\cos\phi + \cancel{\sin\phi} - \cancel{\sin\phi}}{\cancel{\Delta\phi}}$$

$$= \cos\phi$$

Mathe-Crash-Kurs Vorlesung 4

Vorlesung IV

4.10.2013 (1)

Ableitung spezieller Funktionen (Fortsetzung):

$$u(x) = \sin(x), \qquad u'(x) = \cos(x) \tag{1}$$

$$v(x) = \cos(x), \qquad v'(x) = -\sin(x) \tag{2}$$

Bsp. 1

$$t(x) \equiv \tan(x) = \frac{\sin(x)}{\cos(x)}, \qquad t'(x) = \frac{1}{\cos^2(x)} = \sec^2(x)$$

$$\sec(x) \equiv \frac{1}{\cos(x)} = \text{"secans"}$$

Quotientenregel

$$t'(x) = \frac{\cos\cdot\sin' - \sin\cos'}{\cos^2(x)} = \frac{\overbrace{\cos\cdot\cos - \sin(-\sin)}^{=1}}{\cos^2(x)} = \frac{1}{\cos^2(x)} \tag{3}$$

(2)

Bsp. 2

$$\cot(x) \equiv \frac{\cos(x)}{\sin(x)} = \text{"cotangens"}, \qquad \frac{1}{\sin(x)} \equiv \text{cosec}(x) \tag{1}$$

$$\cot'(x) = \frac{\sin(-\sin) - \cos(\cos)}{\sin^2} = \frac{-1}{\sin^2} = -\text{cosec}^2(x) \tag{2}$$

Bsp. 3

$$\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1} \tag{3}$$

Beweisidee:

Ausgangspunkt: $$\frac{d}{dx}\arctan\big(\tan(x)\big) = \frac{d}{dx}x = 1 \tag{4}$$

Kettenregel: $$\arctan'\big(\tan(x)\big)\cdot \underbrace{\tan'(x)}_{(1.3)\; = \frac{1}{\cos^2 x}} = 1 \tag{5}$$

(5) $\cdot \cos^2 x$: $$\arctan'\big(\tan(x)\big) = \cos^2 x \tag{6}$$

Substitution (führe neue Variable ein): $$\tan(x) = y \tag{7}$$

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Nebenrechnung:

$$y = \tan(x) \qquad (1)$$

$$y^2 = \tan^2 = \frac{\sin^2}{\cos^2} = \frac{1-\cos^2}{\cos^2} = \frac{1}{\cos^2} - 1 \qquad (2)$$

$$\frac{1}{\cos^2(x)} = 1 + y^2 \qquad (3)$$

$$\cos^2(x) = \frac{1}{1+y^2} \qquad (4)$$

(2.6):

$$\arctan'(y) = \left(\cos^2(x)\right) \overset{(4)}{=} \frac{1}{1+y^2} = \frac{1}{1+\tan^2(x)}$$

Endergebnis:

$$\arctan'(y) = \frac{1}{1+y^2}$$

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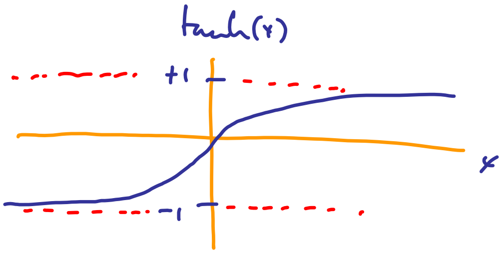
Bsp. 3:

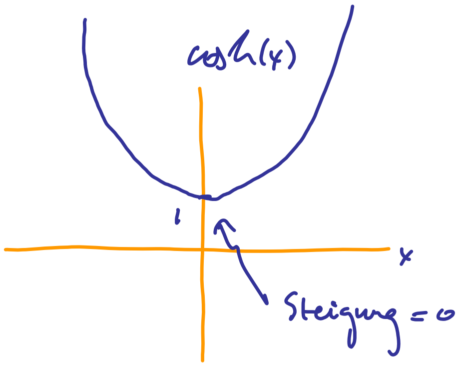
$$f(x) = \tanh(x) \equiv \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad (1)$$

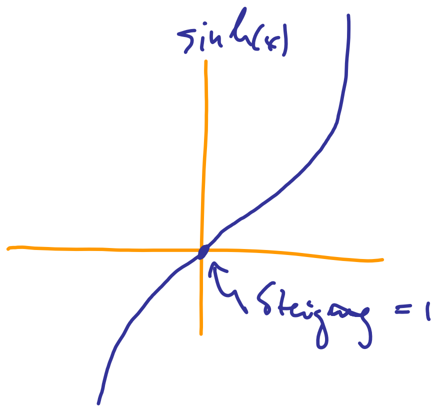
"tangens hyperbolicus"



$$\sinh(x) \equiv \frac{e^x - e^{-x}}{2} \qquad (2)$$

$$\cosh(x) \equiv \frac{e^x + e^{-x}}{2} \qquad (3)$$





$$\sinh'(x) \overset{(4.2)}{\underset{(KR)}{=}} \frac{1}{2}\left(e^{x} - e^{-x}\cdot(-1)\right) \overset{(4.3)}{=} \cosh(x) \qquad (1)$$ ⑤

$$\cosh'(x) \overset{(4.3)}{=} \frac{1}{2}\left(e^{x} + e^{-x}(-1)\right) \overset{(4.2)}{=} \sinh(x) \qquad (2)$$

sehr analog!

Vergleiche:
$$\sin'(x) = \cos(x) \qquad (3)$$
$$\cos'(x) = -\sin(x) \qquad (4)$$

Grund:
$$\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \qquad (5)$$
$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \qquad (6)$$

Also gilt:
$$\sin(x) = \frac{1}{i}\sinh(ix)$$
$$\cos(x) = \cosh(ix)$$

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⑥

$$\tanh'(x) = \frac{d}{dx}\,\frac{\sinh(x)}{\cosh(x)} = \frac{\cosh\cdot\cosh - \sinh\cdot\sinh}{\cosh^2} \qquad (1)$$

$$= \frac{1}{\cosh^2(x)} \equiv \operatorname{sech}^2(x) \quad (2) \qquad \frac{1}{\cosh(x)} \equiv \operatorname{sech}(x) \quad (3)$$

→ "secans hyperbolicus"

Nebenrechnung:

$$\cosh^2(x) - \sinh^2(x) = \frac{1}{4}\left[\left(e^{x}+e^{-x}\right)^2 - \left(e^{x}-e^{-x}\right)^2\right]$$

$$= \frac{1}{4}\left[\left(\cancel{e^{2x}} + 2\cdot\underbrace{e^{x}\cdot e^{-x}}_{=1} + \cancel{e^{-2x}}\right) - \left(\cancel{e^{2x}} - 2\underbrace{e^{x}\cdot e^{-x}}_{=1} + \cancel{e^{-2x}}\right)\right]$$

$$= \frac{1}{4}\left[2+2\right] = 1 = \cosh^2(x) - \sinh^2(x) \qquad (4)$$

(4) ist analogon zu $\cos^2(x) + \sin^2(x) = 1.$ (5)