Back-of-the-Envelope Physics

Winter Term 2022/23

Sheet 8

1. Derive the r-dependence of the following Fourier integrals using dimensional analysis:

$$F_1(r) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{r}} \frac{1}{|\vec{q}|^2}\,, \quad F_2(r) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{r}} \frac{1}{|\vec{q}|}\,, \quad F_3(r) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{r}} \ln(\vec{q}^{\,2}) \tag{1}$$

Check your results by explicit calculations.

2. The scattering amplitude of two heavy masses to one loop in quantum gravity has the schematic form

$$-i\mathcal{M} \sim \frac{GMm}{q^2}\left[1 + aG(M+m)\sqrt{-q^2} + bGq^2\ln(-q^2) + cGq^2\right]\,, \tag{2}$$

where q is the 4-momentum transfer. Obtain the form of the potential $V(r)$ as the Fourier transform of the scattering amplitude in the static limit where $q^2 = -\vec{q}^{\,2}$.

[See *Donoghue (2017), Scholarpedia, 12(4):32997.*]

3. Compute the entropy of an ideal, monatomic gas with energy E, volume V, particle number N and atomic mass m directly from the definition $S = \ln W$ (Sackur-Tetrode equation).

Show that the result can be expressed as

$$S = \frac{3N}{2}\ln\left(\frac{mE}{3\pi\hbar^2}V^{\frac{2}{3}}\right) + \frac{5}{2}N(1 - \ln N) \tag{3}$$

or, equivalently,

$$S = N\left(\ln\frac{V}{N\lambda^3} + \frac{5}{2}\right)\,, \qquad \lambda = \sqrt{\frac{2\pi\hbar^2}{mT}} \tag{4}$$

Discuss the limit $T \to 0$. Give a condition for the validity of the Sackur-Tetrode equation in terms of the density $n = N/V$.

Hint: The D-dimensional solid angle is $\Omega_D = 2\pi^{D/2}/\Gamma(D/2)$.

4. Show that for a set of independent variables (x, y) and a set $(u, v) = (u(x, y), v(x, y))$, the integration measures are related through a functional determinant by

$$du\,dv = \frac{\partial(u, v)}{\partial(x, y)}\,dx\,dy \tag{5}$$