Back-of-the-Envelope Physics

Winter Term 2022/23

Sheet 5

1. Consider the quantum mechanics of a particle of mass m in the 1-dimensional potential $V(r) = \beta r$ for $r > 0$ and $V(r) = \infty$ for $r < 0$.

a) Sketch the potential together with the shape of the ground-state wave function.

b) Use dimensional analysis to find an expression for the energy E_1 of the ground state.

c) Determine the dependence of the energy eigenvalues E_n on the quantum number n (for large n), using the approximation of the potential by an infinite square well of appropriate width.

d) Estimate the asymptotic behaviour of the wave function for large r.

2. The Schrödinger equation for the wave function $u(r)$ of problem 1 is

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \beta r\, u = Eu \tag{1}$$

for $r > 0$, with boundary condition $u(0) = 0$. Obtain the exact solution for the eigenvalues and eigenfunctions. Compare with the approximate results from problem 1.

Hint: Write eq. (1) in terms of a dimensionless variable s through a suitable rescaling of $r = \lambda s$. Similarly, introduce dimensionless energy eigenvalues ε. In this way, eq. (1) can be reduced to the form

$$\frac{d^2u}{dz^2} = zu\,, \tag{2}$$

which is solved by the Airy function

$$\mathcal{A}(z) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + tz\right)\, dt \tag{3}$$

3. The tunneling probability $\mathcal{R}$ for the stellar fusion reaction $p+p \to d+e^{+}+\nu_e$ can be written as $\mathcal{R} = \exp(-\sqrt{E_G/E})$, where E is the energy of the pp collision.

Estimate the Gamow energy E_G for this process, by giving a parametric formula and a numerical evaluation.

4. Including the Boltzmann factor, the tunneling probability from problem 3 becomes $\mathcal{R}_B = \exp(-(\sqrt{E_G/E}+E/T))$. The factor $\mathcal{R}_B(E)$ has a peak at $E = \bar{E}$. Determine the position $\bar{E}(T)$ and the width $\Gamma(T)$ of this peak by expanding the exponent to second order around $\bar{E}$.