Back-of-the-Envelope Physics

Winter Term 2022/23

Sheet 12

1. Consider a sphere of radius $R(t) = a(t)R_0$ and total mass M. Evaluate Newton's second law for a test mass m on the surface of the sphere to find out how the sphere expands or contracts under its own gravity. Defining the Hubble constant via $H \equiv \dot{a}/a$, you should find the Friedmann equation

$$H^2 = \frac{8\pi G\rho}{3} + \frac{2U}{R_0^2 a^2} \tag{1}$$

where $\rho(t)$ is the mass density and U is an integration constant.

2. The current ($t_0 \mathrel{\hat{=}}$ today) value of the Hubble constant is

$$H(t_0) = 68\ \frac{km}{s\ Mpc} \tag{2}$$

Compute the critical mass density $\rho_{crit}(t_0)$ for which $U = 0$. Introducing $\Omega(t) = \rho(t)/\rho_{crit}(t)$, show that the Friedmann equation can be rewritten as

$$1 - \Omega = \frac{2U}{R_0^2 a^2 H^2} \tag{3}$$

Interpret this equation.

3. Derive the fluid equation

$$\dot{\rho} + 3H\rho = 0 \tag{4}$$

4. So far, we have neglected the pressure P. It should come into play when the kinetic energy of the particles due to their uncorrelated thermodynamic movement becomes comparable to their mass-energy density $\epsilon = c^2\rho$. Argue that the fluid equation should then be modified to

$$\dot{\epsilon} + 3H(\epsilon + P) = 0 \tag{5}$$

Do this heuristically by using dimensional analysis and the relativistic energy-momentum relation as well as a bit more rigorously by using the first law of thermodynamics.

5. Combine the Friedmann equation and the fluid equation to obtain the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) \tag{6}$$

6. Using the equation of state $P = w\epsilon$, compute how ϵ scales with a and how a scales with t for the case $U = 0$. Interpret your results.

7. (Difficult) Compute the Friedmann equation and the acceleration equation by evaluating the full Einstein equations for a perfect fluid within a homogeneous and isotropic space and relate U to the spatial curvature.