Neutrino GUT Course

Lecture XXIV

3/2/2023

LMU
Winter 2023

Anomalies (III)

- $$\partial_\mu j_5^\mu = 2 m_f \bar{f} \gamma_5 f +$$

$$+ \frac{g^2}{32\pi^2} F F^d \quad (F\tilde{F})$$

$$F F^d = \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta}$$

$$F F^d = \partial_\mu k^\mu$$

$$\to 0$$

$$K^\mu = \epsilon^{\mu\nu\alpha\beta} \Big(A_\nu^a F_{\alpha\beta}^a +$$

$$+ \_ \, \epsilon_{abc} A_\nu^a A_\alpha^b A_\beta^c \Big)$$

$$F \to 0 \not\Rightarrow A \to 0$$

$$\Downarrow$$

$$\boxed{\Delta B \neq 0 \neq \Delta L \quad (\mathrm{SM})}$$

$$\boxed{\therefore \quad \Gamma \propto e^{-\frac{4\pi}{\alpha}} \to 0}$$

- local (gauge) case

$$\Rightarrow \quad \partial_\mu j^\mu_5 \overset{\downarrow}{=} 0$$ for the sake of renormalizability

$\Downarrow$

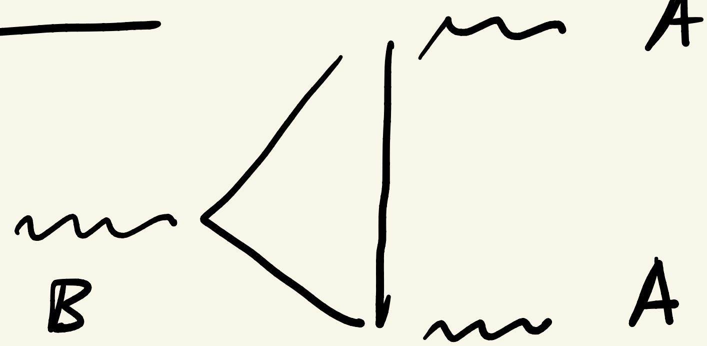
$\boxed{\text{anomalies must cancel}}$

$\Downarrow$

in the SM!

$$\boxed{\begin{aligned} Q_e &= 3\,Q_d \\ Q_\nu &= 0 \end{aligned}}$$

- $\underline{SU(2)^2\,U(1)}$



$$\Rightarrow \mathrm{Tr}\, Y_L = 0 \Rightarrow \mathrm{Tr}\, Q_{em}^{(L)} = 0$$

$$\Rightarrow \boxed{3\, Q_q + Q_\ell = 0}$$

$$q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L \qquad \ell_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_R$$

but: $Q_{em}^{(L)} = Q_{em}^{(R)} \equiv Q_{em}$

$$\Rightarrow \mathrm{Tr}\, Q_{em}^{(R)} = 0$$

$$\Rightarrow \mathrm{Tr}\, Y_R = 0$$

$$(\, Y_U = Y_q + 1 \;, \quad Y_D = Y_q - 1 \,)$$

$$/ \quad Y_E = Y_e - 1 \quad /$$

$$\Updownarrow$$

$$Q_{em} = T_3 + Y/2$$

$$\Downarrow$$

$$3(Y_\ell + 1) + 3(Y_\ell - 1) + Y_e - 1 = 0$$

$$\Rightarrow \quad 6Y_\ell + Y_e - 1 = 0$$

$$\boxed{3Y_\ell + Y_e = 0 \Leftrightarrow 3Q_\ell + Q_e = 0}$$

$$\Downarrow$$

$$3Y_\ell = 1 \Rightarrow Y_\ell = \frac{1}{3}$$

$$\Rightarrow \quad Y_\ell = -1$$

$$\Downarrow$$

$$\Rightarrow \quad \boxed{Q_e = -1, \quad Q_\nu = 0}$$

$$\boxed{\text{Minimal SM} \Rightarrow \text{charge is quantized}}$$

but in minimal SM

$$\Downarrow$$

$$m_\nu = 0$$

- <u>Neutrino mass</u>

$$\exists\, \nu_R \iff \exists\, e_R$$

add: $\boxed{\begin{array}{c} \nu_R{}^T C\, \nu_R \iff \\ Q_\nu = 0 \end{array}}$

$$\Downarrow$$

charge quantization becomes trivial

• imagine no $\nu_R^T C \nu_R$

$$Y(\nu_R) \equiv Y_N \;, \quad Y(e_R) = Y_E$$

$$Y(u_R) \equiv Y_U \;, \quad Y(d_R) = Y_D$$

$$\Downarrow$$

$$Y(\nu_R) = Y(\nu_L)$$

$$Y_N = Y_\ell + 1 \;; \quad Y_E = Y_\ell - 1$$

$$\Updownarrow$$

$$Y_U = Y_\ell + 1 \;; \quad Y_D = Y_\ell - 1$$

$$\Downarrow$$

$$\mathrm{Tr}\, Y_R =$$

$$= 3Y_U + 3Y_D + Y_E + Y_N$$

$$= 3(Y_q + 1 + Y_e - 1) + Y_e - 1 + Y_e + 1$$

$$= 6Y_q + 2Y_e = 2\,\mathrm{Tr}\, Y_L = 0$$

$$\therefore \boxed{3Y_q + Y_e = 0}$$

No complete charge quantization

as in the minimal SM

- Minimal SM (MSM):

$$\boxed{3\, Y_\ell + Y_e = 0}$$

$$Y_e = -1 \Rightarrow Q_\nu = 0$$

$$Y_\phi = +1 \leftarrow \text{Convention}$$

- SM + ν_R:

$$3\, Y_\ell + Y_\ell = 0$$

$$Y_\phi = +1$$

$$Q_e = -\frac{1}{2} + \frac{Y_e}{2} = -\frac{1}{2} + \frac{(-3Y_q)}{2}$$

$$Q_d = -\frac{1}{2} + \frac{Y_q}{2}$$

$$\Downarrow$$

$$Q_e = \frac{-1 - 3Y_e}{2}$$

$$Q_d = \frac{-1 + Y_q}{2}$$

$$\frac{Q_e}{Q_d} = \frac{-1 - 3Y_q}{-1 + Y_q} \neq \text{integer}$$

in general

but in nature: $\frac{Q_e}{Q_d} \simeq 3\left(1 \pm 10^{-20}\right)$

$$\boxed{\text{In MSM : } \quad \frac{Q_e}{Q_d} \overset{\downarrow}{=} 3}$$

• <u>Neutrino mass :</u> Type II

(a) $\underbrace{\ell \quad \Delta \quad \ell}_{C\, i\sigma_2} \Rightarrow Y_\Delta = -2\,Y_\ell$

(b) $\phi \; \Delta^* \; \phi \Rightarrow Y(\Delta^*) = -2\,Y_\phi$

$$\Rightarrow \quad Y_\Delta = 2\,Y_\phi$$

$$\Rightarrow \quad \boxed{Y_\ell = -1}$$ trivial

but $\boxed{\text{neutrino} = \text{Majorana}}$

$\Updownarrow$

$\boxed{\text{type I seesaw}}$

SU(5) anomaly

$\bar{5}_F \; (5_F{}^c) \; , \quad 10_F$

$$\boxed{A(F) = +1} \quad \text{normalization}$$

$$A_{abc} = \mathrm{Tr}\, \{T_a, T_b\}\, T_c =$$

$$= d_{abc}\, A(F)$$

- $f \; , \quad (f^c) \propto f^*$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$T_a \qquad\qquad\qquad -T_a{}^*$$

$$[T_a, T_b] = i f_{abc} T_c$$

$$[-T_a^*, -T_b^*] = i f_{abc} (-T_c^*)$$

$$\Rightarrow \boxed{A(\bar{R}) = -A(R)}$$

$$\Rightarrow \boxed{\begin{aligned} A(\bar{5}_F) &= -1 \\ A(5_F) &= +1 \end{aligned}}$$

$$10 = (5 \times 5)_{AS}$$

$\cdot$ $SU(N)$: $F = N$

$$N \times N = (N \times N)_{AS} + (N \times N)_S$$

$$A(N \times N) = N A(N) + A(N) \times N$$

$$= 2N$$

but

$$A_N(AS) = ?$$

$\bullet$ $SU(3)$

$$\begin{array}{ccccc} 3 \times 3 & = & 6 & + & \underbrace{\overset{???}{3\,(3^*)}}_{} \\ F \times F & = & S & + & AS \end{array}$$

$$(3 \times 3) \times 3 = (6 + 3(3^*)) \times 3$$

$$= 6 \times 3 + \cancel{3}(3^*) \times 3$$

$$\boxed{\text{must be } 3^*}$$

$$(3 \times 3 \times 3)_{AS} = 1_S$$

$$\varepsilon_{ijk} F_i F_j F_k \rightarrow \varepsilon_{ijk} U_{il} U_{jn} U_{kp} \times F_l F_n F_p$$

$$= \varepsilon_{lnp} \overset{1}{\overset{\shortparallel}{(\det U)}} F_l F_n F_p$$

$$= \varepsilon_{ijk} F_i F_j F_k \checkmark$$

$$\Downarrow$$

$$(3 \times 3 \times 3)_{AS} = (3 (3^*) \times 3)_{AS}$$

$$= \text{either } \underbrace{(3 \times 3)_{AS}}_{6 + 3(3^*)} \text{ or } \boxed{\underbrace{(3^* \times 3)_{AS}}_{1_S}}$$

$$F \to UF, \quad F^* \to U^* F^*$$

$$\underline{\underline{F^\dagger F}} \to (F^*)^T F \to F^\dagger U^\dagger U F$$

$$= \underline{\underline{F^\dagger F}}$$

$$\Downarrow$$

$$3 \times 3 = 6 + 3^*$$

$$AS(3\times 3) = 3^*$$

$$\mathcal{A}(AS)_3 = -1 = 3-4$$

anomaly

$\Downarrow$ guess

$$\boxed{\mathcal{A}(AS)_N = N-4}$$

SU(N)

Proof:

Induction: $N=3$ ✓

$$A\,(AS)_{N+1} = ?$$

$$(N+1) \times (N+1) = N \times N + N + N + 1$$

$$\Rightarrow \quad AS_{N+1} = AS_N + N$$

$$S_{N+1} = S_N + N+1$$

<u>Check:</u>

$$AS_N = \frac{N(N-1)}{2}$$

$$S_N = \frac{N(N+1)}{2}$$

$$\Downarrow$$

$$AS_{N+1} = \frac{(N+1)N}{2} = \frac{N^2+N}{2} = \frac{N^2-N+2N}{2}$$

✓

$$= \frac{N^2+N}{2} + N$$

$$S_{N+1} = \frac{(N+1)(N+2)}{2} = \frac{N^2+3N+2}{2}$$

$$= \frac{N(N+1)}{2} + \frac{2N+2}{2}$$

$$= \frac{N(N+1)}{2} + N+1$$

$$\Downarrow$$

$$AS_{N+1} = AS_N + N$$

$$\Downarrow$$

$$\mathbb{A}(AS_{N+1}) = \mathbb{A}(AS_N) + \underset{\substack{\parallel \\ 1}}{\mathbb{A}}(N)$$

$$= N - 4$$

$\uparrow$ by assumption

by normalization

$\Downarrow$

$$\mathbb{A}(AS_{N+1}) = N-4+1 = (N+1)-4$$

$\boxed{Q.E.D}$

$\Downarrow$

$$\mathbb{A}(10_5) = 5-4 = 1$$

$\Downarrow$

$$\mathbb{A}(\bar{5}_F + 10_F) = \mathbb{A}(\bar{5}_F) = -1$$
$$+ \mathbb{A}(10_F) = +1$$

$$\overset{\parallel}{0}$$

$$\Downarrow$$

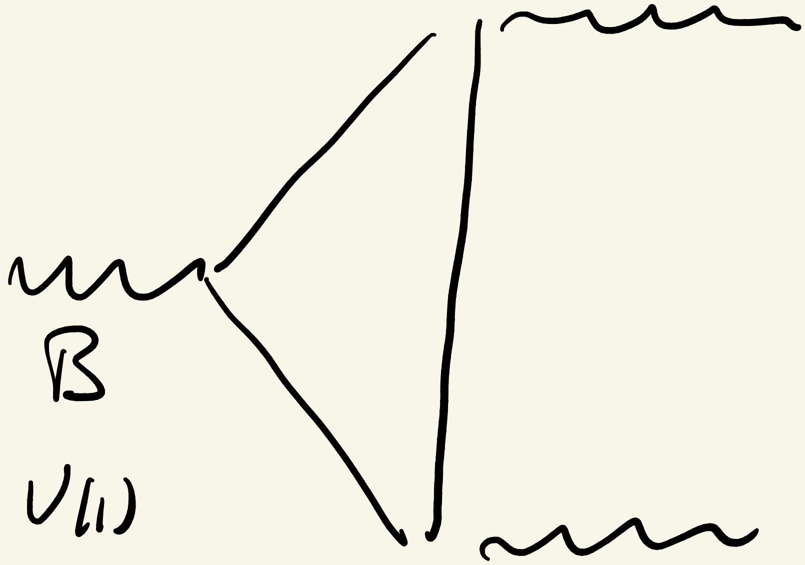
$$\boxed{SU(5) = \text{anomaly free}}$$

$$\bar{5}_F = \begin{pmatrix} d^c \\ \\ \ell_L \end{pmatrix}$$

$$10_F = \begin{pmatrix} u^c & \overbrace{u \quad d}^{Q} \\ & e^c \end{pmatrix}$$

examples !

$U(1)\ SU(2)^2$



$$\boxed{\begin{array}{ll} Y_\ell + Y_\epsilon\, 3 = 0 \\ (\bar{5}_F) \quad (10_F) \end{array}}$$

$$\Updownarrow$$

$$\boxed{\mathrm{Tr}\, Y_L = 0} \Rightarrow \mathrm{Tr}\, Q_L = 0$$

$$\Downarrow \qquad\qquad \Vert$$

$$\boxed{\mathrm{Tr}\, Y_R = 0} \Leftarrow \mathrm{Tr}\, \overset{\Downarrow}{Q_R} = 0$$

$$\underbrace{Y(d^c)}_{\bar{5}_F} + \underbrace{Y(u^c) + Y(e^c)}_{10_F} = 0$$

$\boxed{\bullet \text{ check other anomalies } ***}$

- Recall: charge is quantized

$$Q_\nu = 0 \quad (Q_e = -1)$$

$$\exists\, \nu_R = SU(5) \text{ singlet} = 1_F$$

$$+ \; \bar{5}_F \, 5_H \, 1_F$$

invariant

$\Rightarrow \bar{\nu}_L \, \phi_0 \, \nu_R$

$\Rightarrow \boxed{Q(\nu_R) = Q(\nu_L) = 0}$

SU(5) $\Rightarrow$ all charges are quantized

SM $\Rightarrow$ q, l have charges quantized e.g.

$$D_L, \quad D_R$$ ($SU(2)$ singlets)

$$Y_{D_L} = Y_{D_R}$$

$\uparrow$

arbitrary = <u>not</u> quantized