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Sheet 07: Matrices II: Inverse, Basis Transformation

Posted: Mo 29.11.21 Central Tutorial: Th 02.12.21 Due: Th 09.12.21, 14:00
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 2, 3, 4, 6.
Videos exist for example problems 1 (L5.4.1), 5 (V2.5.1). Also see tutorvideos on "basis transformations".

Example Problem 1: Gaussian elimination and matrix inversion [4]
Points: (a)[1](M); (b)[1](M); (c)[1](M); (d)[1](M)

Gaussian elimination is a convenient book-keeping scheme for solving a linear system of equations of the form $A\mathbf{x} = \mathbf{b}$. For example, consider the system

$$\begin{aligned} A^1{}_1\, x^1 + A^1{}_2\, x^2 + A^1{}_3\, x^3 &= b^1\,, \\ A^2{}_1\, x^1 + A^2{}_2\, x^2 + A^2{}_3\, x^3 &= b^2\,, \\ A^3{}_1\, x^1 + A^3{}_2\, x^2 + A^3{}_3\, x^3 &= b^3\,. \end{aligned}$$

It can be solved by a sequence of steps, each of which involves taking a linear combination of rows, chosen such that the system is brought into the form

$$\begin{aligned} 1\, x^1 + 0\, x^2 + 0\, x^3 &= c^1\,, \\ 0\, x^1 + 1\, x^2 + 0\, x^3 &= c^2\,, \\ 0\, x^1 + 0\, x^2 + 1\, x^3 &= c^3\,. \end{aligned}$$

The solution can then be read off from the right-hand side, $(x^1, x^2, x^3)^T = (c^1, c^2, c^3)^T$.

During these manipulations, time and ink can be saved by refraining from writing down the x^i's over and over again. Instead, it suffices to represent the linear system by an augmented matrix, containing the coefficients in array form, with a vertical line instead of the equal signs. This augmented matrix is to be manipulated in a sequence of steps, each of which involves taking a linear combination of rows, chosen such that the left side is brought into the form of the unit matrix. The right column then contains the desired solution for $(x^1, x^2, x^3)^T$.

$$\begin{array}{ccc|c} x^1 & x^2 & x^3 & \\ \hline A^1{}_1 & A^1{}_2 & A^1{}_3 & b^1 \\ A^2{}_1 & A^2{}_2 & A^2{}_3 & b^2 \\ A^3{}_1 & A^3{}_2 & A^3{}_3 & b^3 \end{array} \quad \longrightarrow \quad \begin{array}{ccc|c} x^1 & x^2 & x^3 & \\ \hline 1 & 0 & 0 & c^1 \\ 0 & 1 & 0 & c^2 \\ 0 & 0 & 1 & c^3 \end{array}$$

Gaussian elimination is also useful for matrix inversion. The inverse of A has the form $A^{-1} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$, where the jth column is the solution of the linear system $A\mathbf{a}_j = \mathbf{e}_j$. The computation of all n vectors $\mathbf{a}_j$ can be done simultaneously by setting up an augmented matrix with n columns on the right, each containing an $\mathbf{e}_j$. After manipulating the augmented matrix such that the left side is the unit matrix, the columns on the right contain the desired vectors $\mathbf{a}_j$.

$$\begin{array}{ccc|ccc} A^1{}_1 & A^1{}_2 & A^1{}_3 & 1 & 0 & 0 \\ A^2{}_1 & A^2{}_2 & A^2{}_3 & 0 & 1 & 0 \\ A^3{}_1 & A^3{}_2 & A^3{}_3 & 0 & 0 & 1 \end{array} \quad \longrightarrow \quad \begin{array}{ccc|ccc} 1 & 0 & 0 & a^1{}_1 & a^1{}_2 & a^1{}_3 \\ 0 & 1 & 0 & a^2{}_1 & a^2{}_2 & a^2{}_3 \\ 0 & 0 & 1 & a^3{}_1 & a^3{}_2 & a^3{}_3 \end{array}$$

(a) Solve the following system of linear equations using Gaussian elimination.

$$\begin{array}{rcrcrcr} 3x^1 & + & 2x^2 & - & x^3 & = & 1, \\ 2x^1 & - & 2x^2 & + & 4x^3 & = & -2, \\ -x^1 & + & \frac{1}{2}x^2 & - & x^3 & = & 0. \end{array}$$

[Check your result: the norm of $\mathbf{x}$ is $\|\mathbf{x}\| = 3$.]

(b) How does the solution change when the last equation is removed?

(c) What happens if the last equation is replaced by $-x^1 + \frac{2}{7}x^2 - x^3 = 0$?

(d) The system of equations given in (a) can also be expressed in the form $A\mathbf{x} = \mathbf{b}$. Calculate the inverse A^{-1} of the 3×3 matrix A using Gaussian elimination. Verify your answer to (a) using $\mathbf{x} = A^{-1}\mathbf{b}$.

Example Problem 2: Two-dimensional rotation matrices [4]

Points: (a)[1](M); (b)[1](E); (c)[1](M); (d)[1](M)

A rotation in two dimensions is a linear map, $R : \mathbb{R}^2 \to \mathbb{R}^2$, that rotates every vector by a given angle about the origin without changing its length.

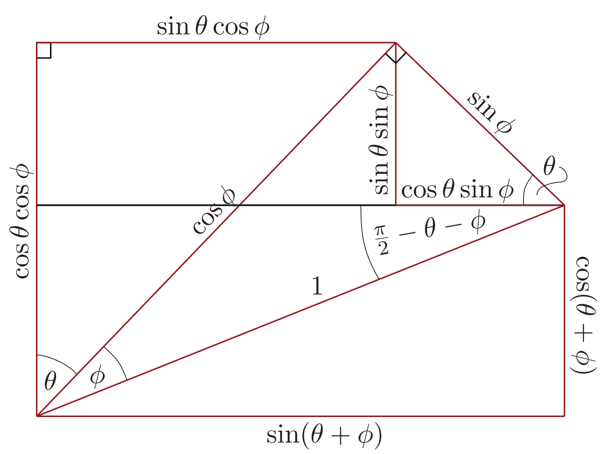
(a) Let the (2×2)-dimensional rotation matrix R_θ describing a rotation by the angle θ be defined by $\mathbf{e}_j \xrightarrow{R_\theta} \mathbf{e}'_j = \mathbf{e}_i (R_\theta)^i{}_j$. Find R_θ by proceeding as follows: Make a sketch that illustrates the effect, $\mathbf{e}_j \xrightarrow{R_\theta} \mathbf{e}'_j$, of the rotation on the two basis vectors $\mathbf{e}_j$ $(j = 1, 2)$ (e.g. for $\theta = \frac{\pi}{6}$). The image vectors $\mathbf{e}'_j$ of the basis vectors $\mathbf{e}_j$ yield the columns of the matrix R_θ.

(b) Write down the matrix R_{θ_i} for the angles $\theta_1 = 0, \theta_2 = \pi/4, \theta_3 = \pi/2$ and $\theta_4 = \pi$. Compute the action of R_{θ_i} $(i = 1, 2, 3, 4)$ on $\mathbf{a} = (1, 0)^T$ and $\mathbf{b} = (0, 1)^T$, and make a sketch to visualize the results.

(c) The composition of two rotations again is a rotation. Show that $R_\theta R_\phi = R_{\theta+\phi}$. *Hint:* Utilize the following 'addition theorems':

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi\,,$$
$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi\,.$$

Remark: A geometric proof of these theorems (not requested here) follows from the figure by inspecting the three right-angled triangles with diagonals of length 1, $\cos\phi$ and $\sin\phi$.



(d) Show that the rotation of an arbitrary vector $\mathbf{r} = (x, y)^T$ by the angle θ does not change its length, i.e. that $R_\theta \mathbf{r}$ has the same length as $\mathbf{r}$.

Example Problem 3: Basis transformations and linear maps in $\mathbb{E}^2$ [4]
Points: (a)[0.5](M); (b)[0.5](M); (c)[0.5](E); (d)[0.5](E); (e)[0.5](M); (f)[0.5](M); (g)[0.5](E); (h)[0.5](E)

Remark on notation: For this problem we denote vectors in Euclidean space $\mathbb{E}^2$ using hats (e.g. $\hat{\mathbf{v}}_j$, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}} \in \mathbb{E}^2$). Their components with respect to a given basis are vectors in $\mathbb{R}^2$ and are written without hats (e.g. $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^2$).

Consider two bases for the Euclidean vector space $\mathbb{E}^2$, one old, $\{\hat{\mathbf{v}}_j\}$, and one new, $\{\hat{\mathbf{v}}'_i\}$, with

$$\hat{\mathbf{v}}_1 = \tfrac{3}{4}\hat{\mathbf{v}}'_1 + \tfrac{1}{3}\hat{\mathbf{v}}'_2 , \quad \hat{\mathbf{v}}_2 = -\tfrac{1}{8}\hat{\mathbf{v}}'_1 + \tfrac{1}{2}\hat{\mathbf{v}}'_2 .$$

(a) The relation $\hat{\mathbf{v}}_j = \hat{\mathbf{v}}'_i T^i{}_j$ expresses the old basis in terms of the new basis. Find the transformation matrix $T = \{T^i{}_j\}$. [Check your result: $\sum_j T^1{}_j = \frac{5}{8}$.]

(b) Find the matrix T^{-1}, and use the inverse transformation $\hat{\mathbf{v}}'_i = \hat{\mathbf{v}}_j (T^{-1})^j{}_i$ to express the new basis in terms of the old basis. [Check your result: $\hat{\mathbf{v}}'_1 - 4\hat{\mathbf{v}}'_2 = -8\hat{\mathbf{v}}_2$.]

(c) Let $\hat{\mathbf{x}}$ be a vector with components $\mathbf{x} = (1, 2)^T$ in the old basis. Find its components $\mathbf{x}'$ in the new basis. [Check your result: $\sum_i x'^i = \frac{11}{6}$.]

(d) Let $\hat{\mathbf{y}}$ by a vector with components $\mathbf{y}' = (\frac{3}{4}, \frac{1}{3})^T$ in the new basis. Find its components $\mathbf{y}$ in the old basis. [Check your result: $\sum_j y^j = 1$.]

(e) Let $\hat{A}$ be the linear map defined by $\hat{\mathbf{v}}'_1 \overset{\hat{A}}{\mapsto} 2\hat{\mathbf{v}}'_1$ and $\hat{\mathbf{v}}'_2 \overset{\hat{A}}{\mapsto} \hat{\mathbf{v}}'_2$. First find the matrix representation A' of this map in the new basis, then use a basis transformation to find its matrix representation A in the old basis. [Check your result: $(A)^2{}_1 = -\frac{3}{5}$.]

(f) Let $\hat{\mathbf{z}}$ be the image vector onto which the vector $\hat{\mathbf{x}}$ is mapped by $\hat{A}$, i.e. $\hat{\mathbf{x}} \overset{\hat{A}}{\mapsto} \hat{\mathbf{z}}$. Find its components $\mathbf{z}'$ with respect to the new basis by using A', and its components $\mathbf{z}$ with respect to the old basis by using A. Are your results for $\mathbf{z}'$ and $\mathbf{z}$ consistent? [Check your result: $\mathbf{z}' = (1, \frac{4}{3})^T$.]

(g) Now make the choice $\hat{\mathbf{v}}_1 = 3\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2$ and $\hat{\mathbf{v}}_2 = -\frac{1}{2}\hat{\mathbf{e}}_1 + \frac{3}{2}\hat{\mathbf{e}}_2$ for the old basis, where $\hat{\mathbf{e}}_1 = (1, 0)^T$ and $\hat{\mathbf{e}}_2 = (0, 1)^T$ are the standard Cartesian basis vectors of $\mathbb{E}^2$. What are the components of $\hat{\mathbf{v}}'_1$, $\hat{\mathbf{v}}'_2$, $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ in the standard basis $\mathbb{E}^2$? [Check your results: $\|\hat{\mathbf{v}}'_1\| = 4$, $\|\hat{\mathbf{v}}'_2\| = 3$, $\|\hat{\mathbf{x}}\| = 2\sqrt{5}$, $\|\hat{\mathbf{z}}\| = 4\sqrt{2}$.]

(h) Make a sketch (with $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ as unit vectors in the horizontal and vertical directions respectively), showing the old and new basis vectors, as well as the vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$. Are the coordinates of these vectors, discussed in (c) and (f), consistent with your sketch?

Example Problem 4: Computing determinants [2]
Points: [2](E)

Compute the determinants of the following matrices by expanding them along an arbitrary row or column. *Hint:* The more zeros the row or column contains, the easier the calculation.

$$A = \begin{pmatrix} 2 & 1 \\ 5 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 & 1 \\ 4 & -3 & 1 \\ 2 & -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} a & a & a & 0 \\ a & 0 & 0 & b \\ 0 & 0 & b & b \\ a & b & b & 0 \end{pmatrix}.$$

[Check your result: if $a = 1$, $b = 2$, then $\det C = -4$.]

Example Problem 5: Jacobian determinant for cylindrical coordinates [2]
Points: (a)[0.5](E); (b)[0.5](E); (c)[1](E).

(a) Compute the Jacobi matrix, $\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}$, for the transformation expressing Cartesian through cylindrical coordinates.

(b) Compute the Jacobi matrix, $J^{-1} = \frac{\partial(\rho,\phi,z)}{\partial(x,y,z)}$, for the inverse transformation expressing cylindrical through Cartesian coordinates. [Check your result: verify that $JJ^{-1} = \mathbb{1}$.]

(c) Compute the Jacobi determinants $\det(J)$ and $\det(J^{-1})$. [Check your results: does their product equal 1?]

Example Problem 6: Triple Gaussian integral via transformation of variables [2]
Points: [2](M)

Calculate the following three-dimensional Gaussian integral (with $a, b, c > 0$, $a, b, c \in \mathbb{R}$):

$$I = \int_{\mathbb{R}^3} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{e}^{-\left[a^2(x+y)^2 + b^2(z-y)^2 + c^2(x-z)^2\right]}.$$

Hint: Use the substitution $u = a(x+y)$, $v = b(z-y)$, $w = c(x-z)$ and calculate the Jacobian determinant, using $J = \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|^{-1}$. You may use $\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{e}^{-x^2} = \sqrt{\pi}$. [Check your result: if $a = b = c = \sqrt{\pi}$, then $I = \frac{1}{2}$.]

Example Problem 7: Variable transformation for two-dimensional integral [Bonus]
Points: (a)[1](E,Bonus); (b)[1](M,Bonus); (c)[1](M,Bonus).

(a) Consider the transformation of variables $x = \frac{1}{2}(X+Y)$, $y = \frac{1}{2}(X-Y)$. Invert it to find $X(x,y)$ and $Y(x,y)$. Compute the Jacobian matrices $J = \frac{\partial(x,y)}{\partial(X,Y)}$ and $J^{-1} = \frac{\partial(X,Y)}{\partial(x,y)}$, and their determinants. [Check your results: verify that $JJ^{-1} = \mathbb{1}$ and $(\det J)(\det J^{-1}) = 1$.]

Use the transformation from (a) to compute the following integrals as $\int \mathrm{d}X\mathrm{d}Y$ integrals:

(b) $I_1 = \int_S \mathrm{d}x\mathrm{d}y$, integrated over the square $S = \{0 \le x \le 1, 0 \le y \le 1\}$.

(c) $I_2(n) = \int_T \mathrm{d}x\mathrm{d}y\,|x-y|^n$, integrated over the triangle $T = \{0 \le x \le 1, 0 \le y \le 1-x\}$. [Check your result: $I_2(1) = \frac{1}{6}$.]

[Total Points for Example Problems: 18]

Homework Problem 1: Gaussian elimination and matrix inversion [3]
Points: (a)[1](E); (b)[1](E); (c)[1](M)

Consider the linear system of equations $A\mathbf{x} = \mathbf{b}$, with

$$A = \begin{pmatrix} 8-3a & 2-6a & 2 \\ 2-6a & 5 & -4+6a \\ 2 & -4+6a & 5+3a \end{pmatrix} . \tag{1}$$

(a) For $a = \frac{1}{3}$, use Gaussian elimination to compute the inverse matrix A^{-1}. (*Remark:* It is advisable to avoid the occurrence of fractions until the left side has been brought into row echelon form.) Use the result to find the solution $\mathbf{x}$ for $\mathbf{b} = (4, 1, 1)^T$. [Check your result: the norm of $\mathbf{x}$ is $\|\mathbf{x}\| = \sqrt{117}/18$.]

(b) For which values of a can the matrix A *not* be inverted?

(c) If A can be inverted, the system of equations $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$, namely $\mathbf{x} = A^{-1}\mathbf{b}$. If A cannot be inverted, then either the solution is not unique, or no solution exists at all — it depends on $\mathbf{b}$ which of these two cases arises. Decide this for $\mathbf{b} = (4, 1, 1)^T$ and the values for a found in (b), and determine $\mathbf{x}$, if possible.

Homework Problem 2: Three-dimensional rotation matrices [4]
Points: (a)[1](E); (b)[1](E); (c)[1](E); (d)[0,5]; (e)[0,5](E); (f)[2](Bonus,A)

Rotations in three dimensions are represented by (3×3)-dimensional matrices. Let $R_\theta(\mathbf{n})$ be the rotation matrix that describes a rotation by the angle θ about an axis whose direction is given by the unit vector $\mathbf{n}$. Its elements are defined via $\mathbf{e}_j \overset{R_\theta(\mathbf{n})}{\longrightarrow} \mathbf{e}'_j = \mathbf{e}_l (R_\theta(\mathbf{n}))^l{}_j$.

(a) Find the three rotation matrices $R_\theta(\mathbf{e}_i)$ for rotations about the three Cartesian coordinate axes $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, by proceeding as follows. Use three sketches, one each for $i = 1, 2, 3$, illustrating the effect, $\mathbf{e}_j \overset{R_\theta(\mathbf{e}_i)}{\longrightarrow} \mathbf{e}'_j$, of a rotation about the i axis on all three basis vectors $\mathbf{e}_j$ $(j = 1, 2, 3)$ (e.g. for $\theta = \frac{\pi}{6}$). The image vectors $\mathbf{e}'_j$ of the basis vectors $\mathbf{e}_j$ yield the columns of $R_\theta(\mathbf{e}_i)$.

(b) It can be shown that for a general direction, $\mathbf{n} = (n_1, n_2, n_3)^T$, of the axis of rotation, the matrix elements have the following form:

$$(R_\theta(\mathbf{n}))^i{}_j = \delta_{ij} \cos\theta + n_i n_j (1 - \cos\theta) - \epsilon_{ijk}\, n_k \sin\theta \qquad (\epsilon_{ijk} = \text{Levi-Civita symbol}) .$$

Use this formula to find the three rotation matrices $R_\theta(\mathbf{e}_i)$ $(i = 1, 2, 3)$ explictly. Are your results consistent with those from (a)?

(c) Write down the following rotation matrices explicitly, and compute and sketch their effect on the vector $\mathbf{v} = (1, 0, 1)^T$:

(i) $A = R_\pi(\mathbf{e}_3)$, (ii) $B = R_{\frac{\pi}{2}}(\frac{1}{\sqrt{2}}(\mathbf{e}_3 - \mathbf{e}_1))$.

(d) Rotation matrices form a group. Use A and B from (c) to illustrate that this group is not commutative (in contrast to the two-dimensional case!).

(e) Show that a general rotation matrix R satisfies the relation $\mathrm{Tr}(R) = 1 + 2\cos\theta$, where the 'trace' of a matrix R is defined by $\mathrm{Tr}(R) = \sum_i (R)^i{}_i$.

(f) The product of two rotation matrices is again a rotation matrix. Consider the product $C = AB$ of the two matrices from (c), and find the corresponding unit vector $\mathbf{n}$ and rotation angle θ. *Hint:* these are uniquely defined only up to an arbitrary sign, since $R_\theta(\mathbf{n})$ and $R_{-\theta}(-\mathbf{n})$ describe the same rotation. (To be concrete, fix this sign by choosing the component n_2 positive.) $|\theta|$ and $|n_i|$ are fixed by the trace and the diagonal elements of the rotation matrix, respectively; their relative sign is fixed by the off-diagonal elements. [Check your result: $n_2 = 1/\sqrt{3}$.]

Homework Problem 3: Basis transformations in $\mathbb{E}^2$ [4]

Points: (a)[0.5](M); (b)[0.5](M); (c)[0.5](E); (d)[0.5](E); (e)[0.5](M); (f)[0.5](M); (g)[0.5](E); (h)[0.5](E)

Remark on notation: For this problem we denote vectors in Euclidean space $\mathbb{E}^2$ using hats (e.g. $\hat{\mathbf{v}}_j$, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}} \in \mathbb{E}^2$). Their components with respect to a given basis are vectors in $\mathbb{R}^2$ and are written without hats (e.g. $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^2$).

Consider two bases for the Euclidean vector space $\mathbb{E}^2$, one old, $\{\hat{\mathbf{v}}_j\}$, and one new, $\{\hat{\mathbf{v}}'_i\}$, with

$$\hat{\mathbf{v}}_1 = \tfrac{1}{5}\hat{\mathbf{v}}'_1 + \tfrac{3}{5}\hat{\mathbf{v}}'_2\,, \quad \hat{\mathbf{v}}_2 = -\tfrac{6}{5}\hat{\mathbf{v}}'_1 + \tfrac{2}{5}\hat{\mathbf{v}}'_2\,.$$

(a) The relation $\hat{\mathbf{v}}_j = \hat{\mathbf{v}}'_i T^i{}_j$ expresses the old basis in terms of the new basis. Find the transformation matrix $T = \{T^i{}_j\}$. [Check your result: $\sum_j T^2{}_j = 1$.]

(b) Find the matrix T^{-1}, and use the inverse transformation $\hat{\mathbf{v}}'_i = \hat{\mathbf{v}}_j (T^{-1})^j{}_i$ to express the new basis in terms of the old basis. [Check your result: $\hat{\mathbf{v}}'_1 + 3\hat{\mathbf{v}}'_2 = 5\hat{\mathbf{v}}_1$.]

(c) Let $\hat{\mathbf{x}}$ be a vector with components $\mathbf{x} = (2, -\frac{1}{2})^T$ in the old basis. Find its components $\mathbf{x}'$ in the new basis. [Check your result: $\sum_i x'^i = 2$.]

(d) Let $\hat{\mathbf{y}}$ by a vector with components $\mathbf{y}' = (-3, 1)^T$ in the new basis. Find its components $\mathbf{y}$ in the old basis. [Check your result: $\sum_j y^j = \frac{5}{2}$.]

(e) Let $\hat{A}$ be the linear map defined by $\hat{\mathbf{v}}_1 \overset{\hat{A}}{\mapsto} \frac{1}{3}(\hat{\mathbf{v}}_1 - 2\hat{\mathbf{v}}_2)$ and $\hat{\mathbf{v}}_2 \overset{\hat{A}}{\mapsto} -\frac{1}{3}(4\hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_2)$. First find the matrix representation A of this map in the old basis, then use a basis transformation to find its matrix representation A' in the new basis. [Check your result: $(A')^2{}_1 = \frac{2}{3}$.]

(f) Let $\hat{\mathbf{z}}$ be the image vector onto which the vector $\hat{\mathbf{x}}$ is mapped by $\hat{A}$, i.e. $\hat{\mathbf{x}} \overset{\hat{A}}{\mapsto} \hat{\mathbf{z}}$. Find its components $\mathbf{z}$ with respect to the old basis by using A, and its components $\mathbf{z}'$ with respect to the new basis by using A'. Are your results for $\mathbf{z}$ and $\mathbf{z}'$ consistent? [Check your result: $\mathbf{z}' = \frac{1}{3}(5, 1)^T$.]

(g) Now make the choice $\hat{\mathbf{v}}_1 = \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2$ and $\hat{\mathbf{v}}_2 = 2\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2$ for the old basis, where $\hat{\mathbf{e}}_1 = (1, 0)^T$ and $\hat{\mathbf{e}}_2 = (0, 1)^T$ are the standard Cartesian basis vectors of $\mathbb{E}^2$. What are the components of $\hat{\mathbf{v}}'_1$, $\hat{\mathbf{v}}'_2$, $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ in the standard basis $\mathbb{E}^2$? [Check your results: $\|\hat{\mathbf{v}}'_1\| = \frac{\sqrt{41}}{4}$, $\|\hat{\mathbf{v}}'_2\| = \frac{\sqrt{89}}{4}$, $\|\hat{\mathbf{x}}\| = \|\hat{\mathbf{z}}\| = \frac{\sqrt{29}}{2}$.]

(h) Make a sketch (with $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ as unit vectors in the horizontal and vertical directions, respectively), showing the old and new basis vectors, as well as the vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$. Are the coordinates of these vectors, discussed in (c) and (f), consistent with your sketch?

Homework Problem 4: Computing determinants [4]
Points: (a)[1](E); (b)[1,5](E); (c)[1,5](E)

(a) Compute the determinant of the matrix $D = \begin{pmatrix} 1 & c & 0 \\ d & 2 & 3 \\ 2 & 2 & e \end{pmatrix}$.

[Check your result: if $c = 1$, $d = 3$, $e = 2$, then $\det C = -2$.]

(i) Which values must c and d have to ensure that $\det D = 0$ for all values of e?

(ii) Which values must d and e have to ensure that $\det D = 0$ for all values of c?

Could you have found the results of (i,ii) *without* explicitly calculating $\det D$?

Now consider the two matrices $A = \begin{pmatrix} 2 & -1 & -3 & 1 \\ 0 & 1 & 5 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 6 & 6 \\ -2 & 8 \\ -2 & -2 \end{pmatrix}$.

(b) Compute the product AB, as well as its determinant $\det(AB)$ and inverse $(AB)^{-1}$.

(c) Compute the product BA, as well as its determinant $\det(BA)$ and inverse $(BA)^{-1}$.

Is it possible to calculate the determinant and the inverse of A and B?

Homework Problem 5: Jacobian determinant for spherical coordinates [2]
Points: (a)[0.5](E); (b)[0.5](E); (c)[1](E).

(a) Compute the Jacobi matrix, $J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$, for the transformation expressing Cartesian through spherical coordinates.

(b) Compute the Jacobi matrix $J^{-1} = \frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$ for the inverse transformation expressing spherical through Cartesian coordinates. [Check your result: verify that $JJ^{-1} = \mathbb{1}$.]

(c) Compute the Jacobi determinants $\det(J)$ and $\det(J^{-1})$. [Check your results: does their product equal 1?]

Homework Problem 6: Triple Lorentz integral via transformation of variables [2]
Points: [2](M)

Calculate the following triple Lorentz integral (with $a, b, c, d > 0$, $a, b, c, d \in \mathbb{R}$):

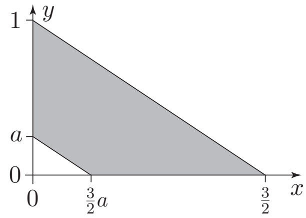
$$I = \int_{\mathbb{R}^3} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \frac{1}{[(xd + y)^2 + a^2]} \cdot \frac{1}{[(y + z - x)^2 + b^2]} \cdot \frac{1}{[(y - z)^2 + c^2]}.$$

Hint: Use the change of variables $u = (xd + y)/a$, $v = (y + z - x)/b$, $w = (y - z)/c$ and compute the Jacobian determinant using $J = \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|^{-1}$. You may use $\int_{-\infty}^{\infty} \mathrm{d}x(x^2 + 1)^{-1} = \pi$. [Check your result: if $a = b = c = \pi$, $d = 2$, then $I = \frac{1}{5}$.]

Homework Problem 7: Variable transformation for two-dimensional integral [Bonus]
Points: (a)[1](E,Bonus); (b)[1](M,Bonus).

(a) Consider the transformation of variables $x = \frac{3}{5}X + \frac{3}{5}Y$ and $y = \frac{3}{5}X - \frac{2}{5}Y$. Invert it to find $X(x,y)$ and $Y(x,y)$. Compute the Jacobian matrices $J = \frac{\partial(x,y)}{\partial(X,Y)}$ and $J^{-1} = \frac{\partial(X,Y)}{\partial(x,y)}$, and their determinants. [Check your results: verify that $JJ^{-1} = \mathbb{1}$ and $(\det J)(\det J^{-1}) = 1$.]

(b) Compute the integral $I(a) = \int_{T_a} \mathrm{d}x\mathrm{d}y\, \cos\big[\pi(\frac{2}{3}x+y)^3\big](x-y)$ over the trapezoid T_a enclosed by the lines $x = 0$, $y = 1 - \frac{2}{3}x$, $y = 0$ and $y = a - \frac{2}{3}x$, with $a \in (0,1)$. *Hint:* Express $I(a)$ as an $\int \mathrm{d}X\mathrm{d}Y$ integral using the transformation from (a).
[Check your result: $I(2^{-1/3}) = -\frac{1}{8\pi}$.]



[Total Points for Homework Problems: 19]