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Sheet 06: Fields II. Matrices I

Posted: Mo 22.11.21 Central Tutorial: Th 25.11.21 Due: Th 02.12.21, 14:00
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Suggestions for central tutorial: example problems 4, 5(bii), 1.
Videos exist for example problems 1 (V3.4.1), 5 (V3.7.3).

Example Problem 1: Potential of a vector field [5]
Points: (a)[1](E); (b)[1](E); (c)[1](M); (d)[1](E);(e)[1](M)

Consider a vector field, $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{r} \mapsto \mathbf{u}(\mathbf{r}) = (2xy + z^3, x^2, 3xz^2)^T$.

(a) Calculate the line integral $I_1 = \int_{\gamma_1} \mathrm{d}\mathbf{r} \cdot \mathbf{u}(\mathbf{r})$ from $\mathbf{0} = (0,0,0)^T$ to $\mathbf{b} = (1,1,1)^T$, along the path $\gamma_1 = \{\mathbf{r}(t) = (t,t,t)^T \mid 0 < t < 1\}$.

(b) Does the line integral depend on the shape of the path?

(c) Calculate the potential $\varphi(\mathbf{r})$ of the vector field $\mathbf{u}(\mathbf{r})$, using the line integral, $\varphi(\mathbf{r}) = \int_{\gamma_\mathbf{r}} \mathrm{d}\mathbf{r} \cdot \mathbf{u}(\mathbf{r})$, along a suitably parametrized path $\gamma_\mathbf{r}$ from $\mathbf{0}$ to $\mathbf{r} = (x,y,z)^T$.

(d) Consistency check: Verify by explicit calculation that your result for $\varphi(\mathbf{r})$ satisfies the equation $\boldsymbol{\nabla}\varphi(\mathbf{r}) = \mathbf{u}(\mathbf{r})$.

(e) Calculate the integral I_1 from part (a) over the vector field by considering the difference in potential $\varphi(\mathbf{r})$ (the antiderivative!) at the integration limits $\mathbf{b}$ and $\mathbf{0}$. Consistency check: Do you obtain the same result as in part (a) of the exercise?

Example Problem 2: Divergence [1]
Points: (a)[E](0,5), (b)[E](0,5).

(a) Compute the divergence, $\boldsymbol{\nabla} \cdot \mathbf{u}$, of the vector field $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{u}(\mathbf{r}) = (xyz, y^2, z^3)^T$.
[Check your results: if $\mathbf{r} = (1,1,1)^T$, then $\boldsymbol{\nabla} \cdot \mathbf{u} = 6$.]

(b) Let $\mathbf{a} \in \mathbb{R}^3$ be a constant vector and $f : \mathbb{R}^3 \to \mathbb{R}$, $\mathbf{r} \mapsto f(r)$ a scalar function of $r = \|\mathbf{r}\|$. Show that

$$\boldsymbol{\nabla} \cdot [\mathbf{a} f(r)] = \frac{\mathbf{r} \cdot \mathbf{a}}{r} f'(r).$$

Rule of thumb: $\boldsymbol{\nabla}$ acting on $f(r)$ generates $\hat{\mathbf{r}} = \mathbf{r}/r$ times the derivative, $f'(r)$.

Example Problem 3: Curl [1]
Points: (a)[E](0,5), (b)[E](0,5).

(a) Compute the curl, $\boldsymbol{\nabla} \times \mathbf{u}$, of the vector field, $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{u}(\mathbf{r}) = (xyz, y^2, z^2)^T$.
[Check your results: if $\mathbf{r} = (3, 2, 1)^T$, then $\boldsymbol{\nabla} \times \mathbf{u} = (0, 6, -3)^T$.]

(b) Let $\mathbf{a} \in \mathbb{R}^3$ be a constant vector and $f : \mathbb{R}^3 \to \mathbb{R}$, $\mathbf{r} \mapsto f(r)$ a scalar function of $r = \|\mathbf{r}\|$. Show that

$$\boldsymbol{\nabla} \times [\mathbf{a} f(r)] = \frac{\mathbf{r} \times \mathbf{a}}{r} f'(r).$$

Rule of thumb: $\boldsymbol{\nabla}$ acting on $f(r)$ generates $\hat{\mathbf{r}} = \mathbf{r}/r$ times the derivative, $f'(r)$.

Example Problem 4: Curl of gradient field [1]
Points: [1](M)

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth scalar field. Show that the curl of its gradient vanishes:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \mathbf{0}.$$

Recommendation: Use Cartesian coordinates, for which contra- and covariant components are equal, $\partial^i = \partial_i$, and write all indices downstairs.

Example Problem 5: Nabla identities [7]
Points: (a)[2](E); (b)[2](M); (c)[3](E)

(a) Consider the scalar fields $f(x, y, z) = z\mathrm{e}^{-x^2}$ and $g(x, y, z) = yz^{-1}$, and the vector fields $\mathbf{u}(x, y, z) = \mathbf{e}_x x^2 y$ and $\mathbf{w}(x, y, z) = (x^2 + y^3)\mathbf{e}_x$. Compute $\boldsymbol{\nabla} f$, $\boldsymbol{\nabla} g$, $\boldsymbol{\nabla}^2 f$, $\boldsymbol{\nabla}^2 g$, $\boldsymbol{\nabla} \cdot \mathbf{u}$, $\boldsymbol{\nabla} \times \mathbf{u}$, $\boldsymbol{\nabla} \cdot \mathbf{w}$, $\boldsymbol{\nabla} \times \mathbf{w}$. [Check your results: at the point $(x, y, z)^T = (1, 1, 1)^T$, we have $\boldsymbol{\nabla} f = (-2\mathrm{e}^{-1}, 0, \mathrm{e}^{-1})^T$, $\boldsymbol{\nabla} g = (0, 1, -1)^T$, $\boldsymbol{\nabla}^2 f = \frac{2}{\mathrm{e}}$, $\boldsymbol{\nabla}^2 g = 2$, $\boldsymbol{\nabla} \cdot \mathbf{u} = 2$, $\boldsymbol{\nabla} \times \mathbf{u} = -\mathbf{e}_z$, $\boldsymbol{\nabla} \cdot \mathbf{w} = 2$, $\boldsymbol{\nabla} \times \mathbf{w} = -3\mathbf{e}_z$.]

(b) Prove the following identities for *general* smooth scalar and vector fields, $f(x, y, z)$, $g(x, y, z)$ and $\mathbf{u}(x, y, z)$, $\mathbf{w}(x, y, z)$. Do *not* represent $\mathbf{u}$, $\mathbf{w}$ and $\boldsymbol{\nabla}$ as column vectors; instead use index notation. *Recommendation:* Use Cartesian coordinates and write all indices downstairs.

(i) $\boldsymbol{\nabla}(fg) = f(\boldsymbol{\nabla} g) + g(\boldsymbol{\nabla} f)$

(ii) $\boldsymbol{\nabla}(\mathbf{u} \cdot \mathbf{w}) = \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{w}) + \mathbf{w} \times (\boldsymbol{\nabla} \times \mathbf{u}) + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{w} + (\mathbf{w} \cdot \boldsymbol{\nabla})\mathbf{u}$

(iii) $\boldsymbol{\nabla} \cdot (f\mathbf{u}) = f(\boldsymbol{\nabla} \cdot \mathbf{u}) + \mathbf{u} \cdot (\boldsymbol{\nabla} f)$

(c) Check the identities from (b) explicitly for the fields given in (a). [Check your results: at the point $(x, y, z)^T = (1, -1, 1)^T$, we have $\boldsymbol{\nabla}(fg) = \mathrm{e}^{-1}(2, 1, 0)^T$, $\boldsymbol{\nabla}(\mathbf{u} \cdot \mathbf{w}) = (-2, -3, 0)^T$, $\boldsymbol{\nabla} \cdot (f\mathbf{u}) = 0$.]

Example Problem 6: Line integral of magnetic field of a current-carrying conductor [4]
Points: (a)[1](E); (b)[1](M); (c)[1](M); (d)[1](E)

This problem illustrates that $\partial_i B^j - \partial_j B^i = 0$ does not necessarily imply $\oint \mathrm{d}\mathbf{r} \cdot \mathbf{B} = 0$.
The magnetic field of an infinitely long current-carrying conductor has the form

$$\mathbf{B}(\mathbf{r}) = \frac{c}{x^2 + y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}.$$

(a) Show that $\partial_i B^j - \partial_j B^i = 0$ holds if $\sqrt{x^2+y^2} \neq 0$.

(b) Compute the line integral $W[\gamma_C] = \int_{\gamma_C} \mathrm{d}\mathbf{r} \cdot \mathbf{B}$ for the closed path along the circle C with radius R around the origin, $\gamma_C = \{\mathbf{r}(t) = R(\cos t, \sin t, 0)^T | t \in [0, 2\pi]\}$.

(c) Compute the line integral $W[\gamma_R] = \int_{\gamma_R} \mathrm{d}\mathbf{r} \cdot \mathbf{B}$ for the closed path γ_R along the edges of the rectangle with corners $(1,0,0)^T$, $(2,0,0)^T$, $(2,3,0)^T$ and $(1,3,0)^T$.

(d) Are your results from (a) to (c) consistent with each other? Explain!

Example Problem 7: Sketching a vector field [Bonus]
Points: (a)[1](M,Bonus); (b)[1](M,Bonus)

Sketch the following vector fields in two dimensions, with $\mathbf{r} = (x, y)^T$:

(a) $\mathbf{u} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \mathbf{r} \mapsto \mathbf{u}(\mathbf{r}) = (\cos y, 0)^T.$

(b) $\mathbf{w} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \mathbf{r} \mapsto \mathbf{w}(\mathbf{r}) = \dfrac{1}{\sqrt{x^2+y^2}}(x, -y)^T.$

For several points $\mathbf{r}$ in the domain of the vector field map (e.g. $\mathbf{u}$), the sketch should depict the corresponding vectors, $\mathbf{u}(\mathbf{r})$, from the codomain of the map. For a chosen point $\mathbf{r}$ one draws an arrow with midpoint at $\mathbf{r}$, whose direction and length represents the vector $\mathbf{u}(\mathbf{r})$. The unit of length may be chosen differently for vectors, $\mathbf{r}$, from the domain and vectors, $\mathbf{u}(\mathbf{r})$, from the codomain, in order to avoid arrows from overlapping and to obtain an uncluttered figure (e.g. by drawing unit vectors $\hat{\mathbf{u}}(\mathbf{r})$ shorter than unit vectors $\hat{\mathbf{r}}$). Indeed, for the visual depiction of codomain vectors usually only their directions and *relative* lengths are of interest, not their absolute lengths.

Example Problem 8: Matrix multiplication [2]
Points: [2](E)

Compute all possible products of pairs of the following matrices, including their squares, where possible:

$$P = \begin{pmatrix} 4 & -3 & 1 \\ 2 & 2 & -4 \end{pmatrix}, \quad Q = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 5 \\ 1 & -6 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} 3 & 0 \\ 1 & 2 \\ 1 & -6 \end{pmatrix}.$$

[Check your results: the sum of all elements of the first column of the following matrix products is: $\sum_i (PQ)^i{}_1 = 14$, $\sum_i (PR)^i{}_1 = 14$, $\sum_i (QR)^i{}_1 = 16$, $\sum_i (RP)^i{}_1 = 12$, $\sum_i (QQ)^i{}_1 = 16$.]

[Total Points for Example Problems: 21]

Homework Problem 1: Line integral of a vector field [2]
Points: [2](M)

Compute the line integral $W[\gamma] = \int_\gamma \mathrm{d}\mathbf{r} \cdot \mathbf{u}$ of the three-dimensional vector field $\mathbf{u}(\mathbf{r}) = (x\mathrm{e}^{yz}, y\mathrm{e}^{xz}, z\mathrm{e}^{xy})^T$ along the straight line γ from the point $\mathbf{0} = (0,0,0)^T$ to the point $\mathbf{b} = b(1,2,1)^T$, with $b \in \mathbb{R}$. [Check your result: for $b^2 = \ln 2$, $W[\gamma] = 7/2$.] Does the line integral depend on the path taken?

Homework Problem 2: Divergence [1]
Points: (a)[E](0,5), (b)[E](0,5).

(a) Compute the divergence, $\nabla \cdot \mathbf{u}$, of the vector field

$$\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3, \quad \mathbf{u}(\mathbf{r}) = (xyz, z^2y^2, z^3y)^T.$$

[Check your results: if $\mathbf{r} = (1,1,1)^T$, then $\nabla \cdot \mathbf{u} = 6$.]

(b) Let $\mathbf{a}$ and $\mathbf{b}$ be constant vectors in $\mathbb{R}^3$. Show that $\nabla \cdot [(\mathbf{a} \cdot \mathbf{r})\,\mathbf{b}] = \mathbf{a} \cdot \mathbf{b}$.

Rule of thumb: ∇ 'kills' the $\mathbf{r}$ in a way that generates another meaningful scalar product.

Homework Problem 3: Curl [1]
Points: (a)[E](0,5), (b)[E](0,5).

(a) Compute the curl, $\nabla \times \mathbf{u}$, of the vector field $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{u}(\mathbf{r}) = (xyz, y^2z^2, xyz^3)^T$.
[Check your result: if $\mathbf{r} = (3,2,1)^T$, then $\nabla \times \mathbf{u} = (-5, 4, -3)^T$.

(b) Let $\mathbf{a}$ and $\mathbf{b}$ be constant vectors in $\mathbb{R}^3$. Show that $\nabla \times [(\mathbf{a} \cdot \mathbf{r})\,\mathbf{b}] = \mathbf{a} \times \mathbf{b}$.

Rule of thumb: ∇ 'kills' the $\mathbf{r}$ in a way that generates another meaningful vector product.

Homework Problem 4: Derivatives of curl of vector field [1]
Points: (a)[1](M), (b)[0,5](M,Bonus), (c)[0,5](E,Bonus).

Let $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. Show that the following identities hold:

(a) $\nabla \cdot (\nabla \times \mathbf{u}) = 0$. (b) $\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}$.

Recommendation: Use Cartesian coordinates and write all indices downstairs.

(c) Check both identities for the field $\mathbf{u}(x,y,z) = (x^2yz, xy^2z, xyz^2)^T$.

Homework Problem 5: Nabla identities [5]
Points: (a)[2](E); (bi,ii)[2](M); (biii)[0,5](M,Bonus); (ci,ii)[1](E); (ciii)[0,5](E,Bonus).

(a) Consider the scalar field $f(x,y,z) = y^{-1}\cos z$ and two vector fields, $\mathbf{u}(x,y,z) = (-y, x, z^2)^T$ and $\mathbf{w}(x,y,z) = (x, 0, 1)^T$. Compute ∇f, $\nabla^2 f$, $\nabla \cdot \mathbf{u}$, $\nabla \times \mathbf{u}$, $\nabla \cdot \mathbf{w}$, $\nabla \times \mathbf{w}$. [Check your results: at the point $(x,y,z)^T = (1,1,0)^T$, $\nabla f = -\mathbf{e}_y$, $\nabla^2 f = 1$, $\nabla \cdot \mathbf{u} = 0$, $\nabla \times \mathbf{u} = 2\mathbf{e}_z$, $\nabla \cdot \mathbf{w} = 1$, $\nabla \times \mathbf{w} = \mathbf{0}$.]

(b) Prove the following identities for *general* smooth scalar and vector fields $f(x,y,z)$, $\mathbf{u}(x,y,z)$ and $\mathbf{w}(x,y,z)$. Do *not* represent $\mathbf{u}$, $\mathbf{w}$ and ∇ as column vectors; instead use index notation. *Recommendation:* Use Cartesian coordinates and write all indices downstairs.

(i) $\nabla \cdot (\mathbf{u} \times \mathbf{w}) = \mathbf{w} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{w})$

(ii) $\nabla \times (f\mathbf{u}) = f\,(\nabla \times \mathbf{u}) - \mathbf{u} \times (\nabla f)$

(iii) $\nabla \times (\mathbf{u} \times \mathbf{w}) = (\mathbf{w} \cdot \nabla)\,\mathbf{u} - (\mathbf{u} \cdot \nabla)\,\mathbf{w} + \mathbf{u}\,(\nabla \cdot \mathbf{w}) - \mathbf{w}\,(\nabla \cdot \mathbf{u})$

(c) Check the identities from (b) explicitly for the fields given in (a).

[Check your results: at the point $(x, y, z)^T = (1, 1, 0)^T$: $\boldsymbol{\nabla} \cdot (\mathbf{u} \times \mathbf{w}) = 2$, $\boldsymbol{\nabla} \times (f\mathbf{u}) = (0, 0, 1)^T$, $\boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w}) = (0, 2, 0)^T$.]

Homework Problem 6: Line integral of vector field on non-simply connected domain [3]

Points: (a)[1](E); (b)[2](M); (c)[2](A,Bonus)

Consider the vector field

$$\mathbf{B}(\mathbf{r}) = \frac{1}{(x^2 + y^2)^2} \begin{pmatrix} -yx^n \\ x^{n+1} \\ 0 \end{pmatrix}.$$

(a) For what value of the exponent n does $\partial_i B^j - \partial_j B^i = 0$ hold, if $\sqrt{x^2 + y^2} \neq 0$?

In the following questions, use the value of n found in (a).

(b) Compute the line integral $W[\gamma_C] = \oint_{\gamma_C} \mathrm{d}\mathbf{r} \cdot \mathbf{B}$ for the closed path along the circle C with radius R around the origin, $\gamma_C = \{\mathbf{r}(t) = R(\cos t, \sin t, 0)^T | t \in [0, 2\pi]\}$.

(c) What is the value of the line integral $W[\gamma_T] = \oint_{\gamma_T} \mathrm{d}\mathbf{r} \cdot \mathbf{B}$ for the closed path γ_T along the edges of the triangle with corners $(-1, -1, 0)^T$, $(1, -1, 0)^T$ and $(a, 1, 0)^T$, with $a \in \mathbb{R}$? Sketch the result as function of $a \in [-2, 2]$. *Hint:* You may write down the result without a calculation, but should offer a justification for it.

Homework Problem 7: Sketching a vector field [Bonus]

Points: (a)[1](M,Bonus); (b)[1](M,Bonus)

Sketch the following vector fields in two dimensions:

(a) $\mathbf{u}(x, y) = (\cos x, 0)^T$, (b) $\mathbf{w}(x, y) = (2y, -x)^T$.

Homework Problem 8: Matrix multiplication [2]

Points: [2](M)

Compute all possible products of pairs of the following matrices, including their squares, where possible:

$$P = \begin{pmatrix} 2 & 0 & 3 \\ -5 & 2 & 7 \\ 3 & -3 & 7 \\ 2 & 4 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} -3 & 1 \\ -1 & 0 \\ 2 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 6 & -1 & 4 \\ 4 & 4 & -4 \\ -4 & -4 & 6 \end{pmatrix}.$$

[Check your results: the sum of all elements of the first column of the following matrix products is: $\sum_i (PQ)^i{}_1 = 25$, $\sum_i (PR)^i{}_1 = -44$, $\sum_i (RQ)^i{}_1 = -5$, $\sum_i (RR)^i{}_1 = 8$.]

[Total Points for Homework Problems: 15]
