Neutrino Mass and Grand Unification

Lecture XXV

28/1/2022

LMU

Winter 2022

Magnetic Monopoles

- $\cancel{\mathbb{Z}_2} \to 1 \Rightarrow$ DW (LR)
- $\cancel{U(1)} \to 1 \Rightarrow$ strings (?)
- GUT : $G \to H \supseteq U(1)$

$$SU(5) \to SU(3) \times SU(2) \times U(1)$$

$$\Downarrow$$

$$\boxed{\text{monopoles}}$$

Simplest example.

$$SU(2) \longrightarrow U(1)$$

$$\Updownarrow$$

$$\boxed{SO(3) \longrightarrow U(1)}$$

Triplet Higgs of $SO(3)$

$\Longleftrightarrow$ Adjoint of $SU(2)$

$$\begin{cases} \phi \ (\vec{\phi}, \ \phi_a \ (a=1,2,3)\ ) = \text{triplet} \\ \phi \longrightarrow O\,\phi \quad \because \ OO^T = O^T O = 1 \\ \qquad\qquad\qquad\qquad \det O = 1 \end{cases}$$

$$\longrightarrow O = e^{i\theta_i T_i} \qquad i = 1,2,3$$

$$\left(= e^{i\theta_{ij} L_{ij}} \qquad L_{ij} = -L_{ji} \qquad i,j = 1,2,3 \right)$$

$$\Rightarrow T_i = -T_i^{*}\,, \qquad T_i = T_i^{\dagger}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} \left(T_i^{(3)}\right)_{jk} &= -i\,\varepsilon_{ijk} \\ [T_i, T_j] &= i\,\varepsilon_{ijk} T_k \end{aligned}}$$

$$T_3^{(3)} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$T_3^{(3)} \longrightarrow \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \quad (\text{"spin" } 1)$$

Adjoint of SU(2)

$$\Sigma \longrightarrow U \Sigma U^{+}$$

$$\Sigma^{+} = \Sigma, \quad \mathrm{Tr}\, \Sigma = 0$$

$$\Sigma = T_a^{(2)} \varphi_a = \frac{\sigma_a}{2} \varphi_a$$

triplet !

$$\Sigma \longrightarrow \left(1 + i \frac{\sigma_i}{2} \theta_i\right) \frac{\sigma_a}{2} \varphi_a \times$$

$$\left(1 - i \frac{\sigma_i}{2} \theta_i\right)$$

$$= \Sigma + i\theta_i \left[\frac{\sigma_i}{2}, \frac{\sigma_a}{2} \right] \varphi_a$$

$$= \Sigma + i\theta_i \, \varepsilon_{iab} \, \frac{\sigma_b}{2} \varphi_a$$

$$= \frac{\sigma_b}{2} \left[\varphi_b + i \theta_i \, \varepsilon_{iab} \varphi_a \right]$$

generators in SO(3)

notation

$\boxed{\text{(complete)}}$

$$\left(T_a^{(3)}\right)_{bc} = i \varepsilon_{abc}$$

$$\mathcal{L} = \frac{1}{2} (D_\mu \phi)_a (D^\mu \phi_a) -$$

$$- \frac{\lambda^2}{2} \left(\phi_a \phi_a - v^2 \right)^2$$

$$- \frac{1}{4} F_{\mu\nu}^{a} F^{\mu\nu a}$$

$$(D_\mu \phi)_a = \left(\partial_\mu - ig\,(T_b A_{\mu b})\right)\phi_a$$

$$= \partial_\mu \phi_a - ig\,(T_b)_{ac} A_{\mu b}\,\phi_c$$

$$= \partial_\mu \phi_a - ig\,(-i\,\varepsilon_{bac} A_{\mu b}\,\phi_c)$$

$$= \partial_\mu \phi_a + g\,\varepsilon_{abc} A_{\mu b}\,\phi_c$$

$\Downarrow$

$$\mathcal{M}_0 = \{\phi_0^a \,:\, \phi_0^a \phi_0^a = v^2\}$$

$$= S_2$$

- $\phi_0^a = v\,\delta_{a3}$ (choice) (Gauge)

$$\Downarrow$$

$$T_3\,\phi_0 = 0$$

$$\phi_a = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \qquad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$SO(3) \xrightarrow[\phi_0]{} U(1)$$

$$\boxed{Q_{em} = T_3}$$

$$(D_\mu \phi)_a = \partial_\mu \phi_a + g\,\varepsilon_{abc} A_\mu \phi_c$$

$$\Downarrow \qquad\qquad (\phi_c = v\,\delta_{c3})$$

$$(D_\mu \phi^0)_a (D^\mu \phi^0)_a =$$

$$= g^2 (\varepsilon_{ab3} A_{\mu b})(\varepsilon_{ad3} A^\mu_{\ d})\, v^2$$

$$m_{A_1} = m_{A_2} = g v$$

$$m_{A_3} = 0$$

$$\Downarrow$$

photon $A^\mu = A_3{}^\mu$, $Q_{em} = T_3$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu =$$

$$= \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3 \neq F_{\mu\nu}^3$$

$$F_{\mu\nu}^3 = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3 + g\, A_\mu^1 A_\nu^2$$

- (ym) $\quad \phi_0^a$, $\quad a \neq 0$

$$Q_{em} = ? \qquad A_\mu = ?$$

$$\phi_a^0 = v\, \delta_{a3} \Rightarrow Q_{em} = T_3$$

guess $\Rightarrow$ $\boxed{Q_{em} = (T_a \phi_a^0)/v}$

$$\Rightarrow (Q_{em}\ \phi_0)_a = (Q_{em})_{ab}\ \phi_0^{\,b}$$

$$= (T_c\ \phi_c^{\,0}/v)_{ab}\ \phi_0^{\,b}$$

$$= (T_c)_{ab}/v\ \ \phi_c^{\,0}\ \phi_0^{\,b}$$

$$= -i\,\varepsilon_{cab}\ \phi_c^{\,0}\ \phi_b^{\,0}/v = 0$$

$$\Downarrow$$

$$\boxed{\begin{array}{c}\text{configuration } \phi_0^{\,a} \\ \Rightarrow\quad Q_{em} = T_a\,\phi_a^{\,0}/v\end{array}}$$

<u>reminder</u>

$$\phi_0^a = v\,\delta_{a3}$$

$$\Rightarrow \quad A_\mu = A_\mu^3$$

Q. Photon for general direction

A. $\boxed{A_\mu^0 = A_\mu^a \frac{\phi_0^a}{v}}$ (PROVE)

<u>Proof</u>

gauge boson mass matrix

$\Rightarrow$ <u>just do it!</u>

$M^2_{ab} \Rightarrow$ Prove that A_μ is massless

- DW : $\mathcal{M}_0 = \{\pm v\}$

$$\Rightarrow \mathcal{M}_\infty = \{+z, -z\}$$

$z \to \infty$

$$\boxed{\phi_{dw}(\pm\infty) = \pm v}$$

- String : $\mathcal{M}_0 = S_1$

$$\Rightarrow \mathcal{M}_\infty = S_1 \text{ (circle} \to \text{large } \rho)$$

$$\phi_{string}(\infty) = v e^{i\theta}$$

- $SO(3) \to U(1)$

$$\mathcal{M}_0 = S_2$$

$$\mathcal{M}_\infty = S_2$$

$$E(\text{top def.}) = \int [\,+\cdots V\,]$$

$E = \text{finite} \Rightarrow V(\text{top. def.}) = 0$
at ∞

$\Rightarrow r, \theta, \varphi$

infinity: $r = R \to \infty$

$$\phi_m^a \, \phi_m^a \xrightarrow[R \to \infty]{} v^2$$

$$V = \frac{\lambda^2}{2} \left(\phi_a \phi_a - v^2 \right)^2$$

$$\vec{\phi}_m \quad \left(\phi_m^a \right)$$

$$\Downarrow$$

$$\left.\begin{aligned}
& && (u) \\
(\phi_m)_1 &\longrightarrow && v \sin\theta \cos\phi \\
(\phi_m)_2 &\longrightarrow && v \sin\theta \sin\phi \\
(\phi_m)_3 &\longrightarrow && v \cos\theta \qquad (u)
\end{aligned}\right\}$$

$$\Downarrow$$

$$\boxed{\vec{\phi}_m \xrightarrow[\infty]{} v\,\hat{r} = v\,\frac{\vec{r}}{r} \quad \Rightarrow \quad \vec{\phi}_m \xrightarrow[R\to 0]{} 0}$$

- $Q_{vac} = T_a \dfrac{\phi_0^a}{v}$

$$\Rightarrow \boxed{Q_m = T_a \frac{\phi_m{}^a}{v}}$$ by analogy

$$(Q_m \phi_m)_a = \left(T_b \frac{\phi_m^b}{v}\right)_{ac} (\phi_m)_c$$

$$= -i\,\varepsilon_{bac}\, \phi_m^b \phi_m^c / v = 0$$

• "photon" $\boxed{A_\mu^{em} = A_\mu^{a} \frac{\phi_m^{a}}{v}}$

$$(D_\mu \phi^{m})_a = \partial_\mu \phi_a^{m} + g\, \epsilon_{abc} A_\mu^{b} \phi_c^{m}$$

$$\phi_m^{a} = v \frac{x_a}{r} \quad (R = \infty)$$

$\Rightarrow$ show that

is a massless eigenstate

$\boxed{\text{Looking for finite E static solution}}$

$$E = \int d^3x \left[\underbrace{\tfrac{1}{2}|D_i\phi|^2}_{\substack{\downarrow \\ 0 \\ (a)}}{}_{R\to\infty} + \underbrace{V(\phi)}_{\substack{\downarrow \\ 0 \\ (b)}} + \cdots \right]$$

(b) $\vec{\phi}_\infty = v\,\hat{r}$: $S_2(M_\infty) \to S_2(M_0)$

$$D_i \vec{\phi}_\infty = 0$$

$$(D_i \phi^\infty)_a = \partial_i \phi^\infty_a - ig(T_b A_i^b)_{ac}\,\phi^\infty_c$$

$$\xrightarrow[R\to\infty]{} 0$$

$$\boxed{\partial_i \phi_m^{\ a} + g\,\epsilon_{abc} A_i^{\ b} \phi_m^{\ c} \underset{(\infty)}{=} 0}$$

∞: $\quad \phi_m^{\ a} = v \dfrac{x_a}{r}$

$$\cancel{v}\left[\frac{\delta_{ia}}{r} + x_a \partial_i \frac{1}{r}\right] +$$

$$+ g\,\epsilon_{abc} A_i^{\ b}\, \cancel{v} \frac{x^c}{r} = 0$$

$$\partial_i \, 1/r = - \frac{x_i}{r^3}$$

$$\boxed{g\,\epsilon_{abc} A_i^{\ b} \frac{x^c}{\cancel{r}} = \frac{\delta_{ia} r^2 - x_i x_a}{r^{\cancel{3}\,2}}} \quad (1)$$

$$\Downarrow$$

$$A_i^{\,b} = \left[a(r)\, \delta_i^{\,b} + b(r)\, \epsilon_{ibc} \frac{x_c}{r} \right]$$

$$\Rightarrow g\, \epsilon_{abc} A_i^{\,b}\, x^c =$$

$$= g\, \epsilon_{abc}\, x_c \left[\cancel{a(r)\, \delta_{ib}} + b(r)\, \epsilon_{ibd} \frac{x_d}{r} \right]$$

$$= g\, \epsilon_{a\underline{b}c}\, \epsilon_{i\underline{b}d} \frac{x_c\, x_d}{r}\, b(r)$$

$$= g \left[\delta_{ai}\, \delta_{cd} - \epsilon_{ad}\, \epsilon_{ci} \right] \frac{x_c\, x_d}{r}\, b(r)$$

$$= g \, \frac{\delta_{ai} r^2 - x_a x_i}{r} \, b(r)$$

$$g \, b(r) = \frac{1}{r}$$

$$\Rightarrow \boxed{A_i^{\,b} = \frac{1}{g} \, \epsilon_{ibc} \, \frac{x_c}{r^2} \qquad R \to \infty}$$

$$\boxed{\bullet \; F_{\mu\nu} = ?}$$

'tHooft

Polyakov 1974

reminder

$$\boxed{\phi_0^a = v\,\delta_{a3} \qquad \phi_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}}$$

$$\Rightarrow \quad A_\mu = A_\mu^3 \qquad \text{but}$$

$$F_{\mu\nu} \neq F_{\mu\nu}^3$$

$$\boxed{F_{\mu\nu} = ?} \quad \Rightarrow \quad F_{\mu\nu} = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$$

$$\neq F_{\mu\nu}^3$$

$\Rightarrow F_{\mu\nu}$ in general case

- $\phi_0^a = v\,\delta_a^3 \Longrightarrow F_{\mu\nu} =$

• for general conf. $\Rightarrow$ gauge invariant

ϕ_a $\Downarrow$ $\left(A_\mu = A_\mu^a \frac{\phi_a}{v} \right)$

$$F_{\mu\nu} = F_{\mu\nu}^a \frac{\phi_a}{v} - \frac{1}{g}\frac{1}{v^3} \epsilon_{abc} (D_\mu \phi)_a (D_\nu \phi)_b \phi_c$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \epsilon_{abc} A_\mu^b A_\nu^c$$

<u>check</u> $\phi_a^0 = v\,\delta_{a3}$ (vacuum)

$\Downarrow$

$$F_{\mu\nu} = F_{\mu\nu}^{\ 3} - \frac{1}{g}\frac{1}{v^2}\epsilon_{ab3}\,(D_\mu\phi)_a(D_\nu\phi)_b$$

$$D_\mu \phi_a = \partial_\mu \phi_a + g\,\epsilon_{amn} A_\mu^m \phi_n$$

$$\Rightarrow \partial_\mu \phi_a^0 = g\,\epsilon_{amn} A_\mu^m v\, \delta_{n3}$$

$$= g\,\epsilon_{am3} A_\mu^m v$$

$$\Downarrow$$

$$F_{\mu\nu} = F_{\mu\nu}^{\ 3} - \frac{1}{g v^2}\left[(D_\mu \phi)_1 (D_\nu \phi_2) - 1 \leftrightarrow 2\right]$$

$$= F_{\mu\nu}^{\ 3} - \frac{\cancel{g}}{g\cancel{v^2}}\left(A_\mu^{1} A_\nu^{2} - A_\mu^{2} A_\nu^{1}\right)\cancel{v^2}$$

$$= \partial_\mu A_\nu^{3} - \partial_\nu A_\mu^{3} \qquad \text{Q.E.D.}$$

Summary

- $\phi_a \Rightarrow \boxed{Q_{em} = T_a \frac{\phi_a}{v}}$

$\therefore \ (Q_{em}\,\phi)_a = 0$

- $\boxed{\text{photon} \quad A_\mu = A_\mu^a \frac{\phi_a}{v}} \ \therefore \ m_A = 0$

$\Downarrow$

- $F_{\mu\nu} = F_{\mu\nu}^a \frac{\phi_a}{v}$ $-$ (not Abelian)

$$- \frac{\varepsilon_{abc}}{g v^3} (D_\mu \phi)_a (D_\nu \phi)_b \phi_c$$

gauge invariant

- $\phi_a = \phi_a^0 \Rightarrow$ trivial

(zero energy)

$m_{A_1} = m_{A_2} = qv$

$m_{A_3} = 0$

$$\boxed{\begin{array}{ccc} \mathcal{M}_\infty & \longrightarrow & \mathcal{M}_0 \\ (S_2) & \longrightarrow & (1 \text{ point}) \end{array}}$$

trivial

- $\phi_a^m = v \frac{x_a}{r}$ $\quad r = R \to \infty$

$F_{\mu\nu}(\phi_a^m) = ?$

A_{ib} = found

Vacuum

$$(D_\mu \phi^0) = 0$$

$$\partial_\mu \phi^0 = 0 \Rightarrow A_\mu^0 = 0$$

$$\Rightarrow F_{\mu\nu}^a (\phi_0) = 0$$

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monopole

$$F_{\mu\nu}(\phi_m) \neq 0$$

$\Rightarrow$ genuine object