

Neutrino Mass  
and  
Grand Unification

Lecture XXV

28/1/2022

LHU  
Winter 2022



# Magnetic Monopoles

- $\mathbb{P}^1 \rightarrow \mathbb{1} \Rightarrow DW (LR)$
- $U(1) \rightarrow \mathbb{1} \Rightarrow \text{string } (?)$
- GUT :  $G \rightarrow H \supseteq U(1)$   
 $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$



monopoles

Simplest example.

$$SU(2) \longrightarrow U(1)$$



$$\boxed{SO(3) \longrightarrow U(1)}$$

Triplet Higgs of  $SO(3)$

$\Leftrightarrow$  Adjoint of  $SU(2)$

$$\left\{ \begin{array}{l} \phi \left( \vec{\Phi}, \phi_a \ (a=1,2,3) \right) = \text{triplet} \\ \phi \longrightarrow O \phi \quad \therefore \quad OO^T = O^T O = 1 \\ \det O = 1 \end{array} \right.$$

$$\rightarrow O = e^{i \theta_i T_i} \quad i = 1, 2, 3$$

$$\left( = e^{i \theta_{ij}} L_{ij} \quad L_{ij} = -L_{ji} \right)$$

$$i, j = 1, 2, 3$$

$$\Rightarrow T_i = -T_i^*, \quad T_c = T_c^\dagger$$

$\Downarrow$

$$\left[ \begin{array}{l} (T_i^{(3)})_{ju} = -i \epsilon_{iju} \\ [T_i, T_j] = i \epsilon_{ijk} T_k \end{array} \right]$$

$$T_3^{(3)} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Downarrow$



$$T_3^{(s)} \rightarrow \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \left( \begin{matrix} \text{"spin"} \\ 1 \end{matrix} \right)$$

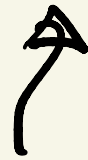

---

Adjoint of SU(2)

$$\Sigma \rightarrow U \Sigma U^\dagger$$

$$\Sigma^\dagger = \Sigma, \quad T_U \Sigma = 0$$

$$\Sigma = T_a^{(2)} \psi_a = \frac{\sigma_a}{2} \psi_a$$



triplet!

$$\bar{\Sigma} \rightarrow \left( 1 + i \frac{\sigma_i}{2} \theta_i \right) \frac{\sigma_a}{2} \psi_a \times$$

$$\left( 1 - i \frac{\sigma_i}{2} \theta_i \right)$$

$$= \Sigma + i \theta_i \left[ \frac{\sigma_i}{2}, \frac{\sigma_a}{2} \right] \varphi_a$$

$$= \Sigma + i \theta_i \varepsilon_{iab} \frac{\sigma_b}{2} \varphi_a$$

$$= \frac{\sigma_b}{2} \left[ \varphi_b + i \theta_i \varepsilon_{iab} \varphi_a \right]$$

(complete)

$$(T_a^{(3)})_{bc} = i \varepsilon_{abc}$$

generators in  $so(3)$   
notation

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} (D_\mu \phi)_a (D^\mu \phi_a) - \\ & - \frac{\lambda^2}{2} (\phi_a \phi_a - v^2)^2 \\ & - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \end{aligned}$$

$$\begin{aligned}
 (D_\mu \phi)_a &= (\partial_\mu - ig (T_b A_{\mu b})) \phi_a \\
 &= \partial_\mu \phi_a - ig (T_b)_{ac} A_{\mu b} \phi_c \\
 &= \partial_\mu \phi_a - ig (-i \epsilon_{bac} A_{\mu b} \phi_c) \\
 &= \partial_\mu \phi_a + g \epsilon_{abc} A_{\mu b} \phi_c
 \end{aligned}$$

$\Downarrow$

$$\begin{aligned}
 M_0 &= \{ \phi_0^a \quad \therefore \quad \phi_0^a \phi_0^a = v^2 \} \\
 &= \mathbb{R}_2
 \end{aligned}$$

$$\cdot \quad \phi_0^a = v \delta_{a3} \quad (\text{choice}) \quad (\text{Goran})$$

$$\Downarrow$$

$$T_3 \phi_0 = 0$$

$$\phi_a = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \quad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\left[ \begin{array}{ccc} SO(3) & \xrightarrow{\phi_0} & U(1) \end{array} \right]$$

$$\boxed{Q_{em} = T_3}$$

$$(D_\mu \phi)_a = \partial_\mu \phi_a + g \sum_{b,c} \epsilon_{abc} A_\mu \phi_c$$

$$\Downarrow \quad \left( \phi_c = v \delta_{c3} \right)$$

$$(D_\mu \phi^0)_a (D^\mu \phi^0)_a =$$


---


$$= g^2 \left( \sum_{b,c} \epsilon_{abc} A_{\mu b} \right) \left( \sum_{d,c} \epsilon_{adc} A^\mu_d \right) v^2$$

$$m_{A_1} = m_{A_2} = g v$$


---


$$m_{A_3} = 0$$

$$\Downarrow$$

photon  $A^\mu = A_3^\mu, \quad Q_{em} = T_3$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu =$$

$$= \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3 \neq F_{\mu\nu}^3$$

$$F_{\mu\nu}^3 = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3 + g \underbrace{A_\mu^1 A_\nu^2}_{\dots\dots\dots}$$

•  $(Y_{em}) \quad \phi_0^a, \quad a \neq 0$

$$Q_{em} = ? \quad A_\mu = ?$$

$$\phi_a^0 = v \delta_{a3} \Rightarrow Q_{em} = T_3$$

$\Rightarrow$   $Q_{em} = (T_a \phi_a^0) / v$

$$\Rightarrow (Q_{em} \phi_0)_a = (Q_{em})_{ab} \phi_0^b$$

$$= (T_c \phi_c^0 / e)_{ab} \phi_0^b$$

$$= (T_c)_{ab} / e \phi_c^0 \phi_0^b$$

$$= -i \epsilon_{cab} \phi_c^0 \phi_b^0 / e = 0$$



configuration  $\phi_0^a$

$$\Rightarrow Q_{em} = T_a \phi_a^0 / e$$

reminder

$$\phi_0^a = v \delta_{a3}$$

$$\Rightarrow A_\mu = A_\mu^3$$

Q. Photon for general direction

A.  $A_\mu^0 = A_\mu^a \frac{\phi_0^a}{v}$  (PROVE)

---

Proof

gauge boson mass matrix

$\Rightarrow$  just do it!



$M_{ab}^2 \Rightarrow$  Prove that  $A_\mu$  is  
massless

---

• DW :  $\mathcal{M}_0 = \{ \pm v \}$   $\rightarrow$   
 $\Rightarrow \mathcal{M}_\infty = \left\{ \begin{array}{c} +z, -z \\ z \rightarrow \infty \end{array} \right\}$   
 $\boxed{\phi_{dw}(\pm\infty) = \pm v}$

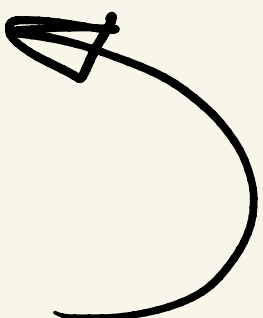
• String :  $\mathcal{M}_0 = S_1$

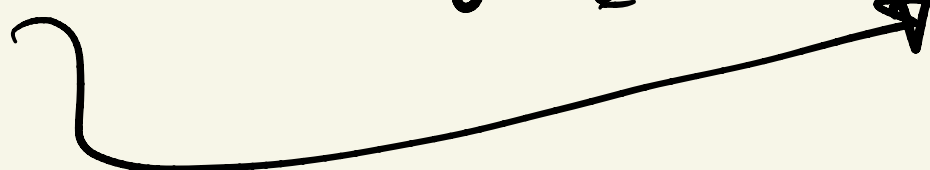
$= \mathcal{M}_\infty = S_1$  (circle  $\rightarrow$  large  $P$ )

$$\phi_{\text{string}}(\theta) = v e^{i\theta}$$

- $SO(3) \rightarrow U(1)$

$$\mathcal{M}_0 = S_2$$

$$\mathcal{M}_\infty = S_2$$


$$E(\text{top def.}) = \int [\dots V]$$


$$E = \text{finite} \Rightarrow V(\text{top. def.}) = 0 \text{ et } \infty$$

$$\Rightarrow 1, 0, 4$$

$$\text{infinity: } V = R \rightarrow \infty$$

$$\phi_m^a \phi_m^a \rightarrow v^2$$

$$R \rightarrow \infty$$

$$V = \frac{\lambda^2}{2} (\phi_a \phi_a - v^2)^2$$

$$\vec{\phi}_m (\phi_m^a)$$

$\Downarrow$

$$\begin{array}{lcl} (\phi_m)_1 & \rightarrow & v \sin \theta \cos \phi \quad \begin{array}{c} (u) \\ \downarrow \end{array} \\ (\phi_m)_2 & \rightarrow & v \sin \theta \sin \phi \quad \downarrow \\ (\phi_m)_3 & \rightarrow & v \cos \theta \quad \begin{array}{c} (u) \end{array} \end{array} \quad \left. \vphantom{\begin{array}{l} (\phi_m)_1 \\ (\phi_m)_2 \\ (\phi_m)_3 \end{array}} \right\}$$

$\Downarrow$

$$\vec{\phi}_m \xrightarrow{r \rightarrow \infty} v \hat{r} = v \frac{\vec{r}}{r}$$

$$\Rightarrow \vec{\phi}_m \xrightarrow{r \rightarrow 0} 0$$

$$\bullet Q_{vac} = T_a \frac{\phi_0^a}{v}$$

$$\Rightarrow \boxed{Q_m = T_a \frac{\phi_m^a}{v}} \quad \text{by analogy}$$

$$(Q_m \phi_m)_a = \left( T_b \frac{\phi_m^b}{v} \right)_{ac} (\phi_m)_c$$

$$= -i \epsilon_{bac} \phi_m^b \phi_m^c / v = 0$$

• "photon"

$$A_\mu^m = A_\mu^a \frac{\phi_m^a}{v}$$

$$(D_\mu \phi^m)_a = \partial_\mu \phi_a^m + g \epsilon_{abc} A_\mu^b \phi_c^m$$

$$\phi_m^a = v \frac{x_a}{r} \quad (r = \infty)$$

$\Rightarrow$  show that

is a massless eigenstate

Looking for finite E  
static solution

$$E = \int d^3x \left[ \underbrace{\frac{1}{2} |\partial_i \phi|^2}_{\substack{R \rightarrow \infty \\ \downarrow \\ 0}} + \underbrace{V(\phi)}_{\substack{\downarrow \\ 0}} \right]$$

(a)                      (b)

$$(b) \quad \vec{\phi}_u = \varphi \vec{r} : S_2(\mathcal{M}_\infty) \rightarrow S_2(\mathcal{M}_0)$$

$$D_i \vec{\phi}_u = 0$$

$$(D_i \phi^u)_a = \partial_i \phi^u_a - ig (T_b A_i^b)_{ac} \phi^u_c$$

$$\xrightarrow{R \rightarrow \infty} 0$$

$$\partial_i \phi_m^a + g \epsilon_{abc} A_i^b \phi_m^c = 0 \quad (\infty)$$

$$\infty: \quad \phi_m^a = v \frac{x_a}{r}$$

$$\cancel{\partial_i \left[ \frac{\delta_{ia}}{r} + x_a \partial_i \frac{1}{r} \right] +}$$

$$+ g \epsilon_{abc} A_i^b \cancel{\frac{x^c}{r}} = 0$$

$$\partial_i \frac{1}{r} = - \frac{x_i}{r^3}$$

$$g \epsilon_{abc} A_i^b \frac{x^c}{r} = \frac{\delta_{ia} r^2 - x_i x_a}{r^3} \quad (1)$$



$$A_i{}^b = \left[ a(\nu) \delta_i{}^b + b(\nu) \epsilon_{ibc} \frac{x_c}{r} \right]$$

$$\Rightarrow g \epsilon_{abc} A_i{}^b x^c =$$

$$= g \epsilon_{abc} x_c \left[ a(\nu) \delta_{ib} + b(\nu) \epsilon_{ibd} \frac{x_d}{r} \right]$$

$$= g \epsilon_{abc} \epsilon_{ibd} \frac{x_c x_d}{r} b(\nu)$$

$$= g \left[ \delta_{ai} \delta_{cd} - \epsilon_{ad} \epsilon_{ci} \right] \frac{x_c x_d}{r} b(\nu)$$



$$= \oint \frac{d\mathbf{a}_i \cdot \mathbf{r} - x_a x_i}{r} b(r)$$

$$g b(r) = \frac{1}{r}$$

$$\Rightarrow \boxed{A_i^b = \frac{1}{g} \epsilon_{ibc} \frac{x_c}{r^2}}$$

$R \rightarrow \infty$

$$\bullet F_{\mu\nu} = ?$$

't Hooft  
Polyakov 1974

reminder

$$\phi_0^a = a f_{a3}$$

$$\phi_0 = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}$$

$$\Rightarrow A_\mu = a_\mu^3 \quad \underline{\text{but}}$$

$$F_{\mu\nu} \neq F_{\mu\nu}^3$$

$$\boxed{F_{\mu\nu} = ?}$$

$$\Rightarrow F_{\mu\nu} = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$$

$$\neq F_{\mu\nu}^3$$

$\Rightarrow F_{\mu\nu}$  in general case

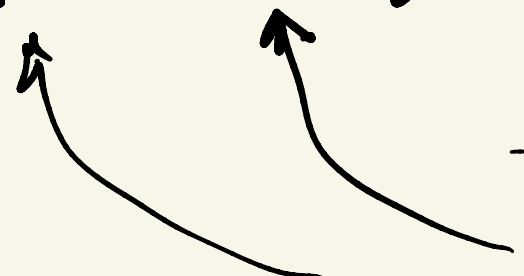
- $\phi_0^a = a f_a^3 \Rightarrow F_{\mu\nu} =$

- for general case.  $\Rightarrow$  gauge invariant

$$\textcircled{\phi_a} \quad \Downarrow \quad (A_\mu = A_\mu^a \frac{\phi_a}{v})$$


---

$$F_{\mu\nu} = F_{\mu\nu}^a \frac{\phi_a}{v}$$

$$= \frac{1}{g} \frac{1}{v^3} \epsilon_{abc} (D_\mu \phi)_a (D_\nu \phi)_b \phi_c$$



---

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \epsilon_{abc} \underset{\uparrow}{A_\mu^b} \underset{\uparrow}{A_\nu^c}$$

check  $\phi_a^0 = v \delta_{a3} \quad (\text{vacuum})$

$$\Downarrow$$

$$F_{\mu\nu} = F_{\mu\nu}^3 - \frac{1}{g} \frac{1}{v^2} \epsilon_{ab3} (D_\mu \phi)_a (D_\nu \phi)_b$$

$$D_\mu \phi_a = \partial_\mu \phi_a + g \epsilon_{am3} A_\mu^m \phi_a$$

$$\Rightarrow \partial_\mu \phi_a^0 = g \epsilon_{am3} A_\mu^m v \delta_{a3}$$

$$= g \epsilon_{am3} A_\mu^m v$$

$\Downarrow$

$$F_{\mu\nu} = F_{\mu\nu}^3 - \frac{1}{g v^2} \left[ (D_\mu \phi)_1 (D_\nu \phi)_2 - 1 \leftrightarrow 2 \right]$$

$$= F_{\mu\nu}^3 - \frac{g}{g v^2} (A_\mu^1 A_\nu^2 - A_\mu^2 A_\nu^1) v^2$$

$$= \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3 \quad \text{Q.E.D.}$$

# Summary

- $\phi_a \Rightarrow \boxed{Q_{em} = T_a \frac{\phi_a}{v}}$

$$\therefore (Q_{em} \phi)_a = 0$$

- $\boxed{\text{photon } A_\mu = A_\mu^a \frac{\phi_a}{v}} \therefore u_A = 0$

---

- $F_{\mu\nu} = F_{\mu\nu}^a \frac{\phi_a}{v} = (\text{not Abelian})$

↗

$$= \frac{\Sigma_{abc}}{g v^3} (D_\mu \phi)_a (D_\nu \phi)_b \phi_c$$

gauge invariant

- $\phi_a = \phi_a^0 \Rightarrow$  trivial  
(zero energy)

$$u_{A_1} = u_{A_2} = qv$$

$$u_{A_3} = 0$$

$$\left[ \begin{array}{ll} \mathcal{M}_\infty & \rightarrow \mathcal{M}_0 \\ (S_2) & \rightarrow (\text{1 point}) \end{array} \right]$$

trivial

- $\phi_a^u = v \frac{x_a}{r} \quad r=R \rightarrow \infty$

$$F_{\mu\nu}(\phi_a^u) = ?$$

$$A_{ib} = \text{found}$$

Vacuum

$$(D_\mu \phi^0) = 0$$

$$\partial_\mu \phi^0 = 0 \Rightarrow A_\mu^0 = 0$$

$$\Rightarrow F_{\mu\nu}^a(\phi_0) = 0$$

---

monopole

$$F_{\mu\nu}(\phi_m) \neq 0$$

$\Rightarrow$  genuine object